The high energy asymptotic distribution of the eigenvalues of the scattering matrix

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Declaration

I certify that the work presented in this thesis is, to the best of my knowledge and belief, original, except as acknowledged in the text, and that the material has not been submitted, either in whole or in part, for a degree at this or any other university.
I acknowledge that I have read and understood the University's rules, requirements, procedures and policy relating to my higher degree research award and to my thesis. I certify that I have complied with the rules, requirements, procedures and policy of the University (as they may be from time to time).

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Abstract

We determine the high energy asymptotic density of the eigenvalues of the scattering matrix associated with the operators $H_0 = -\Delta$ and $H = (i\nabla + A)^2 + V(x)$, where $V : \mathbb{R}^d \to \mathbb{R}$ is a smooth short-range real-valued electric potential and $A = (A_1, \ldots, A_d) : \mathbb{R}^d \to \mathbb{R}^d$ is a smooth short-range magnetic vector-potential. Two cases are considered. The first case is where the magnetic vector-potential is non-zero. The spectral density of the associated scattering matrix in this case is expressed as an integral solely in terms of the magnetic vector-potential A. The second case considered is where the magnetic vector-potential is identically zero. Again the spectral density of the scattering matrix is expressed as an integral, this time in terms of the potential V. These results share similar characteristics to results pertaining to semiclassical asymptotics for pseudodifferential operators.

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1. Introduction

1.1. **Introduction.** We begin by discussing the Schrödinger operators H and H_0 which we shall consider throughout the remainder of this work. Firstly, let us define an operator $\dot{H}_0 = -\Delta$ acting in $L^2(\mathbb{R}^d)$ with domain $C_0^\infty(\mathbb{R}^d)$, where here and throughout we shall assume that $d \geq 2$. Then it is well known that the operator $\dot{H}_0$ is essentially self-adjoint, that is, its closure in $L^2(\mathbb{R}^d)$ is self-adjoint (see e.g. [4] Theorem 3.1.1). We define

$$H_0 = -\Delta$$

to be the closure of $\dot{H}_0$ in $L^2(\mathbb{R}^d)$.

Next, we let $\dot{H} = (i\nabla + A)^2 + V$ be the operator acting in $L^2(\mathbb{R}^d)$ with domain $C_0^\infty(\mathbb{R}^d)$. By V we denote the operator of multiplication by a real-valued electric potential $V : \mathbb{R}^d \to \mathbb{R}$ such that $V \in L^\infty(\mathbb{R}^d)$. By A we denote the multiplication by a magnetic vector-potential $A(x) = (A_1(x), \dots, A_d(x))$ where $A_i : \mathbb{R}^d \to \mathbb{R}$ for all $i = 1, \dots, d$ and $A \in L^\infty(\mathbb{R}^d)$. Then again it is well known that the operator $\dot{H}$ is essentially self adjoint (see e.g. [18] Section X.2), and we define

$$H = (i\nabla + A)^2 + V$$

to be the closure of $\dot{H}$ in $L^2(\mathbb{R}^d)$.

Associated to the pair of operators H and H_0 is an operator $S(k) = S(k; H, H_0)$ known as the scattering matrix (at the energy $k^2 = \lambda > 0$). Under certain assumptions on the potential functions A and V, the scattering matrix is a unitary operator, and further the difference $S(k) - I$ is compact. We briefly discuss here two cases for the potential functions which we shall revisit throughout this work. The first case is where the magnetic vector-potential A is non-zero and V and A are infinitely differentiable satisfying the estimates

$$(1.1) \qquad |\partial^\alpha V(x)| \leq C_\alpha \langle x \rangle^{-\rho-|\alpha|}, \qquad |\partial^\alpha A(x)| \leq C_\alpha \langle x \rangle^{-\rho-|\alpha|}, \qquad \rho > 1,$$

for all multi-indices α; here and throughout $\langle x \rangle = (1+|x|^2)^{1/2}$. In this case the scattering matrix is unitary and the difference $S(k) - I$ is compact - a proof of this fact may be found in [31] Proposition 6.2. The scattering matrix $S(k)$ also depends continuously on k. This point is discussed in Section 1.6.

The second case we consider is when A is identically zero. In this case we assume only that V is continuous and satisfies the short-range condition

$$(1.2) \qquad |V(x)| \leq C\langle x \rangle^{-\rho}, \quad \rho > 1.$$

Again it is well known that the scattering matrix in this case is unitary, that the difference $S(k) - I$ is compact and further that $S(k)$ depends continuously on k (see e.g. [30] Theorem 1.8.1 for details). A simple corollary of the fact that the difference $S(k) - I$ is a compact operator is that the spectrum of the scattering matrix consists solely of eigenvalues of finite multiplicity (except possibly the point 1), which lie on the unit circle and accumulate at the point 1. We label these eigenvalues as

$$\exp(i\theta_n(k)), \quad n \in \mathbb{N}, \quad \theta_n(k) \in [-\pi, \pi). \tag{1.3}$$

The purpose of this study is to determine the asymptotic distribution of the eigenvalues of the scattering matrix at high energy i.e. as $k \to \infty$.

The study of semiclassical asymptotics, and in particular those of the stationary Schrödinger operator

$$-\hbar^2\Delta + V(x)$$

(where $\hbar$ is a semiclassical parameter) is historically well established, with the earliest results in this field dating back more than one hundred years. In the semiclassical limit, i.e. as $\hbar$ becomes very small, one derives a correspondence between the purely quantum objects, such as wave functions and eigenvalues, and the purely classical objects such as the classical trajectories of the associated classical Hamiltonian $H_{\text{class}} = p^2 + V(x)$. Perhaps the most celebrated result in this area, due to H. Weyl [29], is as follows. Let Ω be a bounded region in $\mathbb{R}^d$. We define an operator Δ_D^Ω, called the Dirichlet Laplacian, as the unique self-adjoint operator on $L^2(\Omega)$ whose quadratic form is the closure of the form

$$q(f,g) = \int_\Omega \nabla f \cdot \overline{\nabla g} \mathrm{d}x$$

with domain $C_0^\infty(\mathbb{R}^d)$ (by $\overline{u}$ we denote the complex conjugate of u). Let $N_D(\Omega, \hbar)$ denote the dimension of the range of the spectral projection $P_{[0,1)}$ for $-\hbar^2\Delta_D^\Omega$, that is

$$N_D(\Omega, \hbar) = \dim P_{[0,1)}(-\hbar^2\Delta_D^\Omega).$$

In other words, $N_D(\Omega, \hbar)$ is an eigenvalue counting function for the operator $-\hbar^2\Delta_D^\Omega$. Then Weyl's result states that

$$\lim_{\hbar \to 0} \hbar^d N_D(\Omega, \hbar) = \tau_d (2\pi)^{-d} \operatorname{Vol} \Omega,$$

where τ_d denotes the volume of the unit ball in $\mathbb{R}^d$ and $\operatorname{Vol}\Omega$ denotes the volume of the set Ω. Let us discuss this result. To ease the notation, let us use the transform $m := \hbar^{-2}$, so instead we consider the asymptotics of the above result as $m \to \infty$. Then the asymptotics of $N_D(\Omega, m)$ as $m \to \infty$ is

$$N_D(\Omega, m) \sim (\tau_d m^{d/2}) \operatorname{Vol}\Omega/(2\pi)^d.$$

Hence by informally considering $N_D(\Omega, m)$ as the product of two volumes in $\mathbb{R}^d$, $N_D(\Omega, m)$ is itself associated with a volume in $\mathbb{R}^{2d}$. Indeed, since $(\tau_d m^{d/2}) = \operatorname{Vol}\{p \in \mathbb{R}^d : p^2 < m\}$, it follows that

$$N_D(\Omega, m) \sim \operatorname{Vol}\{(x,p) : x \in \Omega, p^2 < m\}/(2\pi)^d,$$

that is $(2\pi)^d N_D(\Omega, m)$ is asymptotic to the classical phase space volume of a particle moving freely inside Ω with energy $E \le m$. This highlights the correspondence between the purely quantum and purely classical objects in the semiclassical limit. Further, this result agrees with the Bohr-Sommerfeld quantization condition, which states that each quantum state is associated with a volume $(2\pi\hbar)^d = (2\pi m^{-1/2})^d$ in $\mathbb{R}^d$. Weyl's result has far-reaching consequences, for instance it is intrinsic in the famous uncertainty principle. Weyl's result is so well celebrated that results of a similar nature are termed Weyl-type asymptotics.

The more general case of the semiclassical asymptotics of the eigenvalues of the Schrödinger operator with V non-trivial have also been well studied for a multitude of different potentials and in different spaces; see e.g. [10], [20] for many examples of such results. We shall provide one such simple example here to further illustrate the characteristics associated with results of this nature. Consider the Schrödinger operator $-\hbar^2\Delta + V(x)$ acting in $L^2(\mathbb{R}^d)$ and let us suppose that the function $V(x)$ is continuous with a compact support in $\mathbb{R}^d$. We remark here that this result is true for a much wider class of potential functions V, but we have chosen a simple example for illustrative purposes. Let $[a, b] \subset (-\infty, 0)$ be an interval and define the eigenvalue counting function

$$N([a, b]; \hbar) = \dim P_{[a,b]}(-\hbar^2\Delta + V(x)).$$

Then the semiclassical asymptotics of the eigenvalue counting function are given by

$$\lim_{\hbar\to 0} \hbar^d N([a, b]; \hbar) = (2\pi)^{-d} \operatorname{Vol} \Omega_{a,b},$$

where

$$\Omega_{a,b} = \{(p, x) \in \mathbb{R}^d \times \mathbb{R}^d : a \le p^2 + V(x) \le b\}.$$

Again here we see the same characteristics as in the previous result, that is the semiclassical asymptotics of the counting function $N([a, b]; \hbar)$ are associated with a classical phase space volume and the number of quantum states is given by this volume divided by $(2\pi)^d$. The results we shall provide on the distribution of the eigenvalues of the scattering matrix exhibit characteristics of a similar but not identical nature as the those of the results just provided.

1.2. **The purpose of this study.** In this work, we shall study a similar class of problems to those related to semiclassical asymptotics described in Section 1.1, but for the spectrum of the scattering matrix $S(k)$, which has been less well studied. We shall in fact determine the asymptotic distribution of eigenvalues of the scattering matrix $S(k)$ as $k \to \infty$ associated with the magnetic Schrödinger operators $H_0 = -\Delta$, $H = (i\nabla + A)^2 + V(x)$, where $V : \mathbb{R}^d \to \mathbb{R}$ is a real-valued electric potential and $A = (A_1, \dots, A_d) : \mathbb{R}^d \to \mathbb{R}^d$ is a magnetic vector-potential. Our results, similarly to those above, will be expressed in terms of explicit integrals of the potential functions, akin to the classical phase space volumes in the Weyl-type results.

We shall consider two main cases of the above problem. The first case is where the magnetic vector-potential A is non-zero and V and A are infinitely differentiable satisfying the estimates (1.1) for all multi-indices α. Let us define the eigenvalue counting measure for the scattering matrix $S(k)$. We denote by $\operatorname{arc}(t_1, t_2)$ an arc on the unit circle $\mathbb{T}$ of the form

$$\operatorname{arc}(t_1, t_2) = \{e^{i\phi} : t_1 < \phi < t_2\}, \quad (t_1, t_2) \subset (0, 2\pi), \quad t_1 < t_2. \tag{1.4}$$

Then using the notation (1.3), we define the eigenvalue counting measure μ_k for $S(k)$ as

$$\mu_k(\operatorname{arc}(t_1, t_2)) = \#\{n \in \mathbb{N} : \exp(i\theta_n(k)) \in \operatorname{arc}(t_1, t_2)\} \tag{1.5}$$

where # denotes the number of elements in the set. We shall always count the eigenvalues with multiplicities. The weak limit of μ_k as $k \to \infty$ can be expressed in terms of an explicit integral involving the magnetic vector-potential A as follows. Let $\omega \in \mathbb{S}^{d-1}$ and let $\Lambda_\omega \subset \mathbb{R}^d$ be the hyperplane passing through the origin and orthogonal to ω:

$$\Lambda_\omega = \{x \in \mathbb{R}^d : \langle x, \omega \rangle = 0\}.$$

We equip both $\mathbb{S}^{d-1}$ and Λ_ω with the standard $(d-1)$-dimensional Lebesgue measure. We set

$$M(\omega, \xi) = \int_{-\infty}^{\infty} \langle A(t\omega + \xi), \omega \rangle \mathrm{d}t, \quad \omega \in \mathbb{S}^{d-1}, \quad \xi \in \mathbb{R}^d.$$

Let us note here that since $A : \mathbb{R}^d \to \mathbb{R}^d$, the function M is a real-valued function. We define a measure μ^m (here the m denotes magnetic) on $\mathbb{T} \setminus \{1\}$ by

$$\mu^m(\operatorname{arc}(t_1, t_2)) = (2\pi)^{1-d} \int_{\mathbb{S}^{d-1}} \int_{\Pi_\omega(t_1,t_2)} \mathrm{d}\xi \mathrm{d}\omega, \quad 1 \notin (t_1, t_2),$$

where

$$\Pi_\omega(t_1, t_2) = \{\xi \in \Lambda_\omega : e^{iM(\omega,\xi)} \in \operatorname{arc}(t_1, t_2)\}.$$

Then our result states that

$$k^{1-d}\mu_k \to \mu^m \quad \text{weakly as } k \to \infty.$$

This result is stated in Section 1.6.

The second case we shall consider is when A is identically zero, that is $H = -\Delta + V(x)$. We now suppose that V is continuous and satisfies the short-range condition (1.2). In this case, the associated scattering matrix $S(k)$ satisfies the estimate

$$\sup_{k \geq 1} k\|S(k) - I\| \leq C(V).$$

For a proof of the above estimate, see Lemma 3.3. As a consequence, we see that the eigenvalues of $S(k)$ are located on an arc of length $O(k^{-1})$ around 1 as k grows large.

This suggests the following rescaling of the problem: for any interval $(t_1, t_2) \subset \mathbb{R} \setminus \{0\}$ separated away from zero, we set the eigenvalue counting measure $\tilde{\mu}_k$ for $S(k)$ by

$$\tilde{\mu}_k((t_1, t_2)) = \#\{n \in \mathbb{N} : k\theta_n(k) \in (t_1, t_2)\}. \tag{1.6}$$

The weak limit of $\tilde{\mu}_k$ as $k \to \infty$ is again given in terms of an explicit integral but this time depending on the electric potential V. We set

$$X(\omega, \xi) = -\frac{1}{2} \int_{-\infty}^{\infty} V(t\omega + \xi) \mathrm{d}t$$

and define a measure μ^e (here the e denotes electric) on $\mathbb{R} \setminus \{0\}$ by

$$\mu^e((t_1, t_2)) = (2\pi)^{1-d} \int_{\mathbb{S}^{d-1}} \int_{\Gamma_\omega(t_1,t_2)} \mathrm{d}\xi \mathrm{d}\omega, \quad 0 \notin (t_1, t_2),$$

where

$$\Gamma_\omega(t_1, t_2) = \{\xi \in \Lambda_\omega : X(\omega, \xi) \in (t_1, t_2)\}.$$

Then

$$k^{1-d} \tilde{\mu}_k \to \mu^e \quad \text{weakly as } k \to \infty.$$

This result is stated in Section 1.4.

We shall also consider an extension to the second case considered above. We shall study the same problem but with a coupling constant $\alpha > 0$, that is $H_0 = -\Delta$, $H = -\Delta + \alpha V$, where $S(k)$ denotes the scattering matrix associated with these operators. We shall impose the condition

$$\alpha = O(k^\delta), \quad \delta \in [0, 1), \quad k \to \infty. \tag{1.7}$$

In fact this problem is a trivial extension of the problem mentioned above and simply requires different scaling. We discuss this in Section 1.5. We do not look at the borderline case $\delta = 1$ in (1.7) in this work. However we believe a similar result for the case $\delta = 1$ to those stated above can be determined using the results [33] and [34] due to Yafaev by utilising the methods of proof established for the above two main results.

1.3. **Scattering theory.** Scattering theory seeks to describe the results of collisions and interactions of quantum particles after the particles have already diverged a long way from one another and ceased to interact. We shall look at the particular case of particles interacting with potential fields, in particular electric and magnetic potentials. For the case of two particles interacting, one of them (usually the more massive of the two) can be substituted by a potential field over which the other particle is scattered.

Scattering theory may be thought of as a branch of perturbation theory on the absolutely continuous spectrum. We begin with a well understood 'unperturbed' self adjoint operator H_0 acting in a Hilbert space $\mathcal{H}$, which we use to draw conclusions about a more complicated operator H (acting in the same space $\mathcal{H}$), given that H and H_0 differ from one another in a certain sense depending on the particular problem. In a physical description, the operator H_0 is the 'free' Hamiltonian describing a system of non-interacting particles, whereas the full Hamiltonian H describes the 'full' system including all interactions.

In particular, scattering theory concerns itself with solving two related problems. The first of these is understanding the long-time behaviour of solutions of the time dependent Schrödinger equation

$$i\frac{\partial u}{\partial t} = Hu, \quad u(0) = f, \tag{1.8}$$

based on solutions of the time dependent 'free' Schrödinger equation

$$i\frac{\partial u_0}{\partial t} = H_0 u_0. \tag{1.9}$$

The second problem is to determine conditions for the unitary equivalence of the absolutely continuous parts of the operators H and H_0. Clearly, (1.8) has the unique solution

$$u(t) = \exp(-iHt)f.$$

Under certain assumptions on the perturbation $V = H - H_0$, for every f orthogonal to eigenvectors of H, there is a vector $f_0^\pm$ orthogonal to eigenvectors of H_0 such that

$$\lim_{t\to\pm\infty} \|u(t) - u_0(t)\| = 0$$

if $u_0(0) = f_0^\pm$ (here and throughout, the use of $\pm$ refers to two separate identities). For more detail on this point, see for example [30]. In this case, we say that $u(t)$ has free asymptotics as $t \to \pm\infty$. Since from (1.9), $u_0(t) = \exp(-iH_0t)f_0^\pm$, using the previous formula we are led to the following relationship between the inital data f and $f_0^\pm$:

$$f = \lim_{t\to\pm\infty} \exp(iHt)\exp(-iH_0t)f_0^\pm.$$

This motivates the following definition of the wave operators. The definition originated in the physics literature and was defined by Heisenberg and others; we state here the definition given by K .Friedrichs in [11].

Definition 1.1. For a pair of self adjoint operators H and H_0, the wave operators $W_\pm$ are defined by

$$W_\pm = W_\pm(H, H_0) = \operatorname*{s-lim}_{t\to\pm\infty} \exp(iHt)\exp(-iH_0t)P(H_0)$$

provided the corresponding strong limit (s-lim) exists. Here $P(H_0)$ represents the orthogonal projection onto the absolutely continuous subspace $\mathcal{H}_0^{\mathrm{ac}}$ of the operator H_0.

The wave operators are automatically isometric on $\mathcal{H}_0^{\mathrm{ac}}$, and satisfy the intertwining property $HW_\pm = W_\pm H_0$ (see e.g. [4] Section 4.1 for a proof of this result). Thus the range of the wave operators $\operatorname{Ran} W_\pm$ is contained in the absolutely continuous subspace $\mathcal{H}^{\mathrm{ac}}$ of the operator H. This motivates the following definition.

Definition 1.2. The wave operators $W_\pm(H, H_0)$ are said to be complete if

$$\operatorname{Ran} W_+ = \operatorname{Ran} W_- = \mathcal{H}^{\mathrm{ac}}.$$

It can easily be shown that the completeness of the wave operators $W_\pm(H, H_0)$ is equivalent to the existence of the wave operators $W_\pm(H_0, H)$. We provide a proof of this claim shortly, which is based on the following chain rule for the wave operators (for a proof see e.g. [4] Proposition 4.1.9)

Proposition. *Let H_1, H_2, H_3 be self-adjoint operators. If $W_\pm(H_2, H_1)$ and $W_\pm(H_3, H_2)$ exist, then so does the operator $W_\pm(H_3, H_1)$ and*

$$W_\pm(H_3, H_1) = W_\pm(H_3, H_2) W_\pm(H_2, H_1).$$

Let us now prove the above claim. We do this only for the $+$ sign wave operators, it being trivial to replicate the proof for the $-$ sign wave operators. First suppose that both $W_+(H, H_0)$ and $W_+(H_0, H)$ exist. Note that by the chain rule,

$$P(H) = W_+(H, H) = W_+(H, H_0) W_+(H_0, H)$$

where $P(H)$ represents the orthogonal projection onto the absolutely continuous subspace $\mathcal{H}^{\mathrm{ac}}$ of the operator H. Then for $\psi \in \mathcal{H}^{\mathrm{ac}}$,

$$P(H)\psi = W_+(H, H_0)(W_+(H_0, H)\psi)$$

and hence $\mathcal{H}^{\mathrm{ac}} \subset \operatorname{Ran} W_+(H, H_0)$ so that $W_+(H, H_0)$ is complete.
Now suppose that $W_+(H, H_0)$ is complete. Let $\psi \in \mathcal{H}^{\mathrm{ac}}$, then there exists some $\phi \in \mathcal{H}_0^{\mathrm{ac}}$ such that $W_+(H, H_0)\phi = \psi$. Hence

$$\lim_{t \to \infty} \| e^{itH} e^{-itH_0} \phi - \psi \| = 0,$$

and since $e^{itH} e^{-itH_0}$ is a unitary operator it follows that

$$\lim_{t \to \infty} \| \phi - e^{itH_0} e^{-itH} \psi \| = 0.$$

This is sufficient to conclude that $W_+(H, H_0)$ exists, and by repeating the above for the wave operator $W_-(H, H_0)$, the above claim is proved. $\square$
Consequently, if $W_\pm(H, H_0)$ exist and are complete, then the absolutely continuous parts of the operators H and H_0 are unitarily equivalent.
Let us now describe the existence and completeness of wave operators for the two specific cases we shall study. The first result is due to T. Kato [13].

Proposition 1.3. *Let $H_0 = -\Delta$ and let $H = -\Delta + V$ be the Schrödinger operators acting in $L^2(\mathbb{R}^d)$, $d \geq 2$. Suppose that V is the operator of multiplication by a real-valued potential which satisfies the short-range condition* (1.2)*. Then the wave operators $W_\pm(H, H_0)$ exist and are complete.*

The second result is well-known (see e.g. [30] Theorem 1.10.2).

Proposition 1.4. *Let $H_0 = -\Delta$ and let $H = (i\nabla + A)^2 + V(x)$ be the Schrödinger operators acting in $L^2(\mathbb{R}^d)$, where $A = (A_1, \ldots, A_d) : \mathbb{R}^d \to \mathbb{R}^d$ is a magnetic vector-potential and $V : \mathbb{R}^d \to \mathbb{R}$ is a real-valued electric potential. Suppose that the functions V, A satisfy the short-range condition* (1.2). *Then the wave operators $W_\pm(H, H_0)$ exist and are complete.*

Via the wave operators, we define another important object in scattering theory called the scattering operator. The scattering operator is defined as

$$\mathbf{S} = \mathbf{S}(H, H_0) = W_+^* W_-.$$

The above operator connects the 'free' asymptotics of a quantum system as $t \to \pm\infty$, as seen from the relation

$$\mathbf{S} f_0^- = f_0^+.$$

Thus the scattering operator allows us to consider complicated interactions by understanding the 'initial' and 'final' characteristics of the simpler free problem. For perturbations V satisfying the short-range condition (1.2), the scattering operator $\mathbf{S}$ is unitary in $\mathcal{H} = L^2(\mathbb{R}^d)$, $d \geq 2$, and the commutation relation

$$\mathbf{S} H_0 = H_0 \mathbf{S} \tag{1.10}$$

holds (see e.g. [4] Proposition 4.1.7). We now discuss the standard spectral representation of the self-adjoint operator H_0 in the space $\mathfrak{h} = L^2(\mathbb{R}_+; L^2(\mathbb{S}^{d-1}))$, which is the direct integral of identical spaces $L^2(\mathbb{S}^{d-1})$ enumerated by the numbers $k^2 = \lambda > 0$. We define a unitary operator $\Gamma_0 : \mathcal{H} \to \mathfrak{h}$ by

$$(\Gamma_0 u)(k; \omega) = 2^{-1/2} k^{\frac{d-2}{2}} \widehat{u}(k\omega), \quad u \in C_0^\infty(\mathbb{R}^d), \quad \omega \in \mathbb{S}^{d-1}, \tag{1.11}$$

where we understand $(\Gamma_0 u)(k; \omega)$ as a function of k with values in $L^2(\mathbb{S}^{d-1})$ and where $\widehat{u}$ is the usual (unitary) Fourier transform of u, that is

$$\hat{u}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\langle \xi, x\rangle} u(x) \mathrm{d}x, \quad \xi \in \mathbb{R}^d.$$

It is clear that we have the following diagonalisation of the operator H_0;

$$(\Gamma_0 H_0 u)(k; \omega) = k^2 (\Gamma_0 u)(k; \omega), \quad \forall u \in C_0^\infty(\mathbb{R}^d).$$

The commutation relation (1.10) implies that $\Gamma_0(k)$ also diagonalises the scattering operator $\mathbf{S}$, i.e.

$$(\Gamma_0 \mathbf{S} u)(k; \omega) = \tilde{\mathbf{S}} (\Gamma_0 u)(k; \omega)$$

where $\tilde{\mathbf{S}}$ is the operator of multiplication by the scattering matrix $S(k)$ in the space $\mathfrak{h} = L^2(\mathbb{R}_+; L^2(\mathbb{S}^{d-1}))$.

We end this section by introducing the stationary representation of the scattering matrix (for more details see Section 3 and e.g. [30] Section 8.1). Let the potential V satisfy the short range estimate (1.2). We shall factorize V as

$$V = \langle x\rangle^{-\rho/2} J \langle x\rangle^{-\rho/2}$$

where $J = J(\rho)$ is the bounded operator of multiplication by $\langle x\rangle^{\rho} V(x)$. By $T(z)$ we denote the sandwiched resolvent operator

$$T(z) = \langle x\rangle^{-\rho/2}(H - zI)^{-1}\langle x\rangle^{-\rho/2}, \quad \operatorname{Im} z > 0.$$

Next, for $k > 0$ and $\rho > 1$ we define an operator $\Gamma_\rho(k) : L^2(\mathbb{R}^d) \to L^2(\mathbb{S}^{d-1})$ by

$$(\Gamma_\rho(k)u)(\omega) = (\Gamma_0(\langle x\rangle^{-\rho/2}u))(k;\omega), \quad u \in C_0^\infty(\mathbb{R}^d), \quad \omega \in \mathbb{S}^{d-1},$$

where Γ_0 is defined in (1.11). Then the scattering matrix $S(k)$ may be expressed in the stationary representation as

$$S(k) - I = -2\pi i \Gamma_\rho(k)[J - JT(k^2 + i0)J]\Gamma_\rho(k)^*.$$

Let us denote the resolvent operators of H and H_0 by

$$R(z) = (H - zI)^{-1}, \quad R_0(z) = (H_0 - zI)^{-1}, \quad \operatorname{Im} z > 0.$$

Then as a result of the resolvent identity

$$R(z) = R_0(z) - R_0(z)VR(z) = R_0(z) - R(z)VR_0(z), \quad \operatorname{Im} z > 0,$$

and the estimate (3.3), it may be shown the scattering matrix can be written as the following asymptotic expansion (the Born expansion):

$$S(k) - I = -2\pi i \sum_{n=0}^{N} (-1)^n \Gamma_\rho(k)[JT_0(k^2 + i0)]^n J\Gamma_\rho(k)^* + G_N(k)$$

where for any $N \in \mathbb{N}$, the operator $G_N(k)$ satisfies

$$\|G_N(k)\| = O(k^{-N+2}), \quad k \to \infty.$$

The first term of this expansion i.e.

$$S_B(k) := -2\pi i \Gamma_\rho(k) J \Gamma_\rho(k)^* \tag{1.12}$$

is referred to as the Born approximation to the scattering matrix.

1.4. **The case of an electric potential.** Let $H_0 = -\Delta$ and let $H = -\Delta + V$ be the Schrödinger operators in $L^2(\mathbb{R}^d)$, $d \geq 2$, where V is the operator of multiplication by a continuous real-valued potential $V : \mathbb{R}^d \to \mathbb{R}$, which satisfies the short-range condition (1.2). By $S(k)$ we denote the scattering matrix associated with the operators H and H_0. The scattering matrix $S(k)$ is continuous in k as are the scattering phases $\theta_n(k)$ defined in (1.3). Let us now explain the definition of the eigenvalue counting measure $\tilde{\mu}_k$ for $S(k)$, as defined in (1.6) (some of the material which follows was introduced in Section 1.2 but we shall restate it here for completeness).

We begin by recalling the estimate (see Lemma 3.3)

$$\|S(k) - I\| = O(k^{-1}), \quad k \to \infty. \tag{1.13}$$

This estimate is sharp, i.e. $O(k^{-1})$ cannot be replaced by $o(k^{-1})$; this can be seen by considering the case of a spherically symmetric potential and using the separation of variables. Thus, the spectrum of the scattering matrix $S(k)$ for large k consists of a cluster of eigenvalues located on an arc of length $O(k^{-1})$ around 1. This explains the scaling by k in the definition of $\tilde{\mu}_k$, and also why the measure is defined on the real-line as opposed to the seemingly more natural domain of definition $\mathbb{T}$.

We now describe the weak limit of $\tilde{\mu}_k$ as $k \to \infty$ as follows. For any $\omega \in \mathbb{S}^{d-1}$, let $\Lambda_\omega \subset \mathbb{R}^d$ denote the hyperplane passing through the origin and orthogonal to ω. We equip both $\mathbb{S}^{d-1}$ and Λ_ω with the standard $(d-1)$-dimensional Lebesgue measure (=Euclidean area). We set

$$X(\omega, \xi) = -\frac{1}{2}\int_{-\infty}^{\infty} V(t\omega + \xi)\mathrm{d}t, \quad \omega \in \mathbb{S}^{d-1}, \quad \xi \in \Lambda_\omega. \tag{1.14}$$

The function X (up to a multiplicative factor) is known as the X-ray transform of V in the inverse problem literature. The following elementary estimate is a direct consequence of (1.2):

$$|X(\omega, \xi)| \le C(V)(1 + |\xi|)^{1-\rho}, \quad \omega \in \mathbb{S}^{d-1}, \quad \xi \in \Lambda_\omega, \tag{1.15}$$

with some constant $C(V)$. Let us prove (1.15). Let us begin by writing

$$X(\omega, \xi) = -\frac{1}{2}\int_{-\infty}^{\infty} V(t\omega + \xi)\langle t\omega + \xi\rangle^{\rho}\langle t\omega + \xi\rangle^{-\rho}\mathrm{d}t, \quad \rho > 1.$$

Note that from the estimate (1.2) we obtain

$$\max_{x \in \mathbb{R}^d} |V(x)\langle x\rangle^{\rho}| < +\infty,$$

and hence

$$|X(\omega, \xi)| \le \left(\max_{x \in \mathbb{R}^d} |V(x)\langle x\rangle^{\rho}|\right)\int_{-\infty}^{\infty} \langle t\omega + \xi\rangle^{-\rho}\mathrm{d}t. \tag{1.16}$$

Using the estimate

$$\langle x\rangle^{-\rho} \le C(1 + |x|)^{-\rho}, \quad \rho > 1,$$

together with the Pythagorean identity for orthogonal vectors

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2, \quad \langle x, y\rangle = 0,$$

we easily see that the right hand side of (1.16) can be estimated above by

$$C(V)\int_{0}^{\infty} (t + (1 + |\xi|))^{-\rho}\mathrm{d}t.$$

Evaluating the above integral shows that

$$|X(\omega,\xi)| \le C(V)(1+|\xi|)^{1-\rho},$$

as required.

We define a measure μ^e on $\mathbb{R}\setminus\{0\}$ by

$$\mu^e((t_1,t_2)) = (2\pi)^{1-d}\int_{\mathbb{S}^{d-1}}\int_{\Gamma_\omega(t_1,t_2)} \mathrm{d}\xi\mathrm{d}\omega, \quad 0\notin(t_1,t_2), \tag{1.17}$$

where

$$\Gamma_\omega(t_1,t_2) = \{\xi\in\Lambda_\omega : X(\omega,\xi)\in(t_1,t_2)\}.$$

By the boundedness of V, the measure μ^e has a compact support. The measure μ^e need not be absolutely continuous. The measure μ^e may be weakly singular at zero in the following sense: $\mu^e((0,\infty))$ or $\mu^e((-\infty,0))$ may be infinite, but, by the estimate (1.15) we have

$$\int_{-\infty}^{\infty}|t|^\ell \mathrm{d}\mu^e(t) < \infty, \quad \forall \ell > (d-1)/(\rho-1). \tag{1.18}$$

Indeed, the left hand side of (1.18) can be written as

$$\int_{\mathbb{S}^{d-1}}\int_{\Lambda_\omega} |X(\omega,\xi)|^\ell \mathrm{d}\xi\mathrm{d}\omega, \quad \omega\in\mathbb{S}^{d-1}, \quad \xi\in\Lambda_\omega,$$

which is finite by the estimate (1.15) for all $\ell > (d-1)/(\rho-1)$.

The main result of this section is as follows:

Theorem 1.5. *Let $V:\mathbb{R}^d\to\mathbb{R}$ be a continuous potential satisfying the short-range condition* (1.2)*. Then for the measures $\tilde\mu_k$ and μ^e defined in* (1.6) *and* (1.17) *respectively,*

$$k^{1-d}\tilde\mu_k \to \mu^e \quad \textit{weakly as} \quad k\to\infty. \tag{1.19}$$

More explicitly, we may state (1.19) as follows: for any test function $\varphi\in C_0^\infty(\mathbb{R}\setminus\{0\})$,

$$\lim_{k\to\infty} k^{1-d}\int_{-\infty}^{\infty}\varphi(t)\mathrm{d}\tilde\mu_k(t) = \int_{-\infty}^{\infty}\varphi(t)\mathrm{d}\mu^e(t).$$

Remark 1.6. In the paper [5], the case of potentials with the power asymptotics at infinity of the type

$$V(x) = v(x/|x|)|x|^{-\rho}(1+o(1)), \quad |x|\to\infty, \quad \rho>1,$$

was considered. It is assumed that the potential $V\in L^\infty(\mathbb{R}^d)$ and $v\in C^\infty(\mathbb{S}^{d-1})$. Using the notation $\tilde\mu_k,\mu^e$, the result of [5] can be written as

$$k^{1-d}\tilde\mu_k((\varepsilon,\infty)) \sim \mu^e((\varepsilon,\infty)),$$
$$k^{1-d}\tilde\mu_k((-\infty,-\varepsilon)) \sim \mu^e((-\infty,-\varepsilon))$$

when $k > 0$ is fixed and $\epsilon \to +0$. Here $a \sim b$ means $\frac{a}{b} \to 1$. In other words, the above result determines the asymptotic distribution of the scattering phases $\theta_n(k)$ as $n \to \infty$ with k fixed. In the paper [5], their result was also determined by considering asymptotics of a pseudodifferential operator but in a different asymptotic regime. Clearly the result (1.19) is expressed by the same formula as above. However, since uniform asymptotics of pseudodifferential operators are not available, neither our result nor the result of [5] is uniform in the other variable (n or k) and so neither implies the other.

Remark 1.7 (Semiclassical interpretation). By the definition of the scattering operator $\mathbf{S}$, for any $\psi \in L^2(\mathbb{R}^d)$ we have

$$i((\mathbf{S}-I)\psi,\psi) = i\lim_{t\to\infty}((e^{-2itH}e^{itH_0}\psi, e^{-itH_0}\psi) - \|\psi\|^2) = i\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}(e^{-2itH}e^{itH_0}\psi, e^{-itH_0}\psi)\mathrm{d}t$$
$$= \int_0^\infty (Ve^{-2itH}e^{itH_0}\psi, e^{-itH_0}\psi)\mathrm{d}t + \int_0^\infty (Ve^{itH_0}\psi, e^{2itH}e^{-itH_0}\psi)\mathrm{d}t.$$

Recall that the potential function V satisfies the short-range estimate (1.2). Therefore if ψ corresponds to large energies, the significant contribution to the total energy is determined by the kinetic energy. In terms of our operators, this means that the operator H_0 dominates the potential energy given by $V(x)$, at least in quadratic form terms. As a consequence of this and recalling $H = H_0 + V$, the right hand side of the above can be approximated by the first term in its expansion in powers of V. This means that we can replace e^{itH} by e^{itH_0} in the above expressions, and so

$$i((\mathbf{S}-I)\psi,\psi) \approx \int_{-\infty}^\infty (Ve^{-itH_0}\psi, e^{-itH_0}\psi)\mathrm{d}t, \quad \psi \in L^2(\mathbb{R}^d), \tag{1.20}$$

which is exactly the Born approximation (1.12) in the time-dependent picture.

In order to write down the classical analogue of the right hand side of (1.20), assume that ψ is concentrated near x in the coordinate representation and near p in the momentum representation. Then ψ represents a particle with the coordinate x and momentum p, and in the same way $e^{-itH_0}\psi$ represents a particle with the coordinate $x + 2pt$ and momentum p. Thus, the classical analogue of the right hand side of (1.20) is

$$\int_{-\infty}^\infty V(x+2pt)\mathrm{d}t = \frac{1}{2|p|}\int_{-\infty}^\infty V(x+\omega t')\mathrm{d}t', \quad (x,p) \in \mathbb{R}^d \times \mathbb{R}^d,$$

where $\omega = \frac{p}{|p|} \in \mathbb{S}^{d-1}$. Note that the above is an integral over the free dynamics. This calculation explains the appearance of the X-ray transform in the spectral asymptotics of $S(k)$.

1.5. **The case of a coupling constant.** Let $H_0 = -\Delta$ and let $H = -\Delta + \alpha V$ be the Schrödinger operators in $L^2(\mathbb{R}^d)$, $d \geq 2$ where $V : \mathbb{R}^d \to \mathbb{R}$ is a continuous electric potential satisfying the short-range condition (1.2) and $\alpha > 0$ is a coupling constant. By $S(k)$ we denote the scattering matrix associated with the operators H and H_0. We impose upon the coupling constant α the condition

$$\alpha = O(k^\delta), \quad \delta \in [0,1), \quad k \to \infty. \tag{1.21}$$

The problem constitutes a trivial extension of Theorem 1.5. Similarly to the estimate (1.13), we have

$$\|S(k) - I\| = O(k\alpha^{-1}), \quad \alpha = O(k^\delta), \quad \delta \in [0,1), \quad k \to \infty. \tag{1.22}$$

A proof of this estimate may be found in Lemma 3.3. The estimate (1.22) motivates the following definition of the eigenvalue counting measure $\tilde{\mu}_k$:

$$\tilde{\mu}_k((t_1,t_2)) = \#\{n \in \mathbb{N} : (k\alpha^{-1})\theta_n(k) \in (t_1,t_2)\}. \tag{1.23}$$

The weak limit of $\tilde{\mu}_k$ is given in the following result.

Theorem 1.8. *Let $V : \mathbb{R}^d \to \mathbb{R}$ be a continuous electric potential satisfying the short-range condition* (1.2) *and let the coupling constant $\alpha = O(k^\delta)$, $\delta \in [0,1)$ as $k \to \infty$. Then for the measures $\tilde{\mu}_k$ and μ^e defined in* (1.23) *and* (1.17) *respectively,*

$$k^{1-d}\tilde{\mu}_k \to \mu^e \quad \textit{weakly as} \quad k \to \infty. \tag{1.24}$$

1.6. **The case of a magnetic potential.** Let $H_0 = -\Delta$ and let $H = (i\nabla + A)^2 + V$ be the Schrödinger operators in $L^2(\mathbb{R}^d)$, $d \geq 2$. Here $V : \mathbb{R}^d \to \mathbb{R}$ is an electric potential and $A = (A_1, \ldots, A_d) : \mathbb{R}^d \to \mathbb{R}^d$ is a magnetic vector-potential. We assume that both V and A are infinitely differentiable and satisfy the estimates (1.1) for all multi-indices α. By $S(k)$ we denote the scattering matrix associated with the operators H and H_0. As mentioned in Section 1.1, the operator $S(k)$ is unitary and the difference $S(k) - I$ is compact. Further, the scattering matrix $S(k)$ depends continuously on k. In order to see this, we note that in the work [12], the absence of positive eigenvalues for the operator $H = (i\nabla + A)^2 + V$ is established, where A and V need only satisfy the short-range condition (1.2) with $A \in C^1(\mathbb{R}^d)$ and $V \in C(\mathbb{R}^d)$. Using this fact and following the scheme set out in [30] Theorem 1.8.1 and its preceding discussion, in this case one can determine the continuity of the scattering matrix in the variable k.

Let us briefly remark here that the scattering matrix $S(k)$ is gauge invariant in the following sense. Let A be a magnetic vector-potential which is differentiable and satisfies the short-range condition

$$|A(x)| \leq C\langle x\rangle^{-\rho}, \quad \rho > 1.$$

We define a function

$$\widetilde{A}(x) = A(x) + \nabla\phi(x) \tag{1.25}$$

where $\phi(x) \in C^1(\mathbb{R}^d)$ satisfies the estimate

$$\lim_{|x|\to\infty} |\nabla\phi(x)| = 0. \tag{1.26}$$

Then the scattering matrix $\widetilde{S}(k)$ associated with the pair $\widetilde{H} = (i\nabla + \widetilde{A})^2 + V$, $H_0 = -\Delta$, coincides with $S(k)$. See [21, 32] for further details.

We now describe the weak limit of the eigenvalue counting measure μ_k for $S(k)$ as defined in (1.5). We set

$$M(\omega, \xi) = \int_{-\infty}^{\infty} \langle A(t\omega + \xi), \omega \rangle \mathrm{d}t, \quad \omega \in \mathbb{S}^{d-1}, \quad \xi \in \Lambda_\omega, \tag{1.27}$$

where Λ_ω denotes the hyperplane passing through the origin and orthogonal to ω as before. Note that it is easily seen that M is gauge invariant under the gauge transformations of the class (1.25) where ϕ satisfies (1.26). Indeed, we need only show

$$\int_{-\infty}^{\infty} \langle \nabla\phi(t\omega + \xi), \omega \rangle \mathrm{d}t = 0, \quad \omega \in \mathbb{S}^{d-1}, \quad \xi \in \Lambda_\omega.$$

By definition,

$$\int_{-\infty}^{\infty} \langle \nabla\phi(t\omega + \xi), \omega \rangle \mathrm{d}t = \int_{-\infty}^{\infty} \left[\frac{\mathrm{d}}{\mathrm{d}t}\phi(t\omega + \xi) + \left\langle \hat{\xi}\frac{\partial}{\partial \xi}\phi(t\omega + \xi), \omega \right\rangle \right] \mathrm{d}t,$$

where $\hat{\xi} = \xi|\xi|^{-1}$. Then since $\langle \hat{\xi}, \omega \rangle = 0$,

$$\int_{-\infty}^{\infty} \langle \nabla\phi(t\omega + \xi), \omega \rangle \mathrm{d}t = \lim_{T \to +\infty} [\phi(T\omega + \xi) - \phi(-T\omega + \xi)] = 0$$

as required.

The following elementary estimate is a direct consequence of (1.1):

$$|M(\omega, \xi)| \le C(A)(1 + |\xi|)^{1-\rho}, \quad \omega \in \mathbb{S}^{d-1}, \quad \xi \in \Lambda_\omega. \tag{1.28}$$

The estimate (1.28) is derived in an almost identical way to that of (1.15). Using the notation (1.4), let us define a measure μ^m on $\mathbb{T} \setminus \{1\}$ by

$$\mu^m(\mathrm{arc}(t_1, t_2)) = (2\pi)^{1-d} \int_{\mathbb{S}^{d-1}} \int_{\Pi_\omega(t_1, t_2)} \mathrm{d}\xi \mathrm{d}\omega, \quad 1 \notin \mathrm{arc}(t_1, t_2), \tag{1.29}$$

where

$$\Pi_\omega(t_1, t_2) = \{\xi \in \Lambda_\omega : e^{iM(\omega, \xi)} \in \mathrm{arc}(t_1, t_2)\}.$$

The measure μ^m may be weakly singular at the point $z = 1$ in the following sense: $\mu^m(\mathbb{T} \setminus \{1\})$ may be infinite, but by the estimate (1.28) we have

$$\int_{\mathbb{T}} |z - 1|^\ell \mathrm{d}\mu^m(z) < +\infty, \quad \forall \ell > (d-1)/(\rho - 1). \tag{1.30}$$

The main result of this section is as follows:

Theorem 1.9. *Let $V : \mathbb{R}^d \to \mathbb{R}$ and let $A : \mathbb{R}^d \to \mathbb{R}^d$ both be infinitely differentiable and satisfy the estimates* (1.1)*. Then for the measures μ_k and μ^m defined in* (1.5) *and* (1.29) *respectively,*

$$k^{1-d}\mu_k \to \mu^m \quad \textit{weakly as} \quad k \to \infty. \tag{1.31}$$

More explicitly, the result (1.31) may be expressed as follows: for any $\varphi \in C^\infty(\mathbb{T}\setminus\{1\})$ that vanishes in a neighbourhood of the point 1,

$$\lim_{k\to\infty} k^{1-d}\operatorname{Tr}\varphi(S(k)) = (2\pi)^{1-d}\int_{\mathbb{S}^{d-1}}\int_{\Lambda_\omega}\varphi(e^{iM(\omega,\xi)})\mathrm{d}\xi\mathrm{d}\omega.$$

In fact, Theorem 1.9 holds true for a wider class of functions than mentioned above. For example, by following the proof, it is easy to see that Theorem 1.9 holds true for any function $\varphi \in C(\mathbb{T})$ such that $\varphi(z)|z-1|^{-\ell_0}$ is continuous, where ℓ_0 is the smallest even integer greater than $(d-1)/m$, $m = \min\{1, \rho-1\}$.

Remark 1.10. Our main interest for Theorem 1.9 is in the cases $d = 2, 3$. In dimension $d = 3$, a magnetic vector-potential A satisfying (1.1) can be constructed for any smooth magnetic field $B : \mathbb{R}^3 \to \mathbb{R}^3$ ($B = \operatorname{curl} A$) such that $\operatorname{div} B = 0$ and

$$|\partial^\alpha B(x)| \le C_\alpha \langle x\rangle^{-\rho-1-|\alpha|}, \quad \rho > 1. \tag{1.32}$$

Let us provide details of one such construction (see e.g. [32] for details). Firstly we define a magnetic vector-potential $A^{\mathrm{tr}}(x) = (A_1^{\mathrm{tr}}(x), A_2^{\mathrm{tr}}(x), A_3^{\mathrm{tr}}(x))$ which satisfies the short range condition

$$|A^{\mathrm{tr}}(x)| \le C\langle x\rangle^{-\rho}, \quad \rho > 1,$$

and the transversal gauge condition

$$\langle A^{\mathrm{tr}}(x), x\rangle = 0.$$

This is defined by the formula

$$A_1^{\mathrm{tr}}(x) = \int_0^1 (B_2(sx)x_3 - B_3(sx)x_2)s\mathrm{d}s$$

and the components A_2^{tr} and A_3^{tr} are defined by cyclic permutations of the above. For a magnetic field satisfying (1.32), then $A^{\mathrm{tr}}(x)$ admits the representation

$$A^{\mathrm{tr}}(x) = A^\infty(x) + A^{\mathrm{reg}}(x)$$

where $A^\infty(x) = (A_1^\infty(x), A_2^\infty(x), A_3^\infty(x))$, $A^{\mathrm{reg}}(x) = (A_1^{\mathrm{reg}}(x), A_2^{\mathrm{reg}}(x), A_3^{\mathrm{reg}}(x))$ and

$$A_1^\infty(x) = |x|^{-2}\int_0^\infty (B_2(s\hat{x})x_3 - B_3(s\hat{x})x_2)s\mathrm{d}s,$$

$$A_1^{\mathrm{reg}}(x) = -|x|^{-2}\int_{|x|}^\infty (B_2(s\hat{x})x_3 - B_3(s\hat{x})x_2)s\mathrm{d}s.$$

Again the remaining components of $A^\infty(x)$ and $A^{\mathrm{reg}}(x)$ are obtained by cyclic permutations of the above formulae. Then $A^\infty(x)$ is a homogeneous function of degree -1 which satisfies the following two equations:

$$\operatorname{curl} A^\infty(x) = 0,$$

$$\langle A^\infty(x), x\rangle = 0.$$

The function $A^{\mathrm{reg}}(x)$ satisfies the short-range condition

$$|A^{\mathrm{reg}}(x)| \leq C\langle x\rangle^{-\rho}, \quad \rho > 1.$$

Given a magnetic field B as defined above, we now construct a magnetic vector-potential $A(x)$ satisfying (1.1). Define the function $U(x)$ for $x \neq 0$ as the curvilinear integral

$$U(x) = \int_{\Gamma_{x_0,x}} \langle A^\infty(y), \mathrm{d}y\rangle \tag{1.33}$$

taken between some fixed point $x_0 \neq 0$ and x, where $\Gamma_{x_0,x}$ is a contour which does not contain 0. Clearly

$$A^\infty(x) = \operatorname{grad} U(x)$$

and further it can be shown that $U(x)$ is homogeneous of degree 1. Now choose $R_2 > R_1 > 0$ and a function $\eta \in C^\infty(\mathbb{R}^3)$ such that $\eta(x) = 1$ for $|x| \geq R_2$ and $\eta(x) = 0$ for $|x| \leq R_1$. Then $A(x)$ is constructed as

$$A(x) = A^{\mathrm{tr}}(x) - \operatorname{grad}(\eta(x)U(x))$$

which satisfies (1.1) as required.

In dimension $d = 2$, if the magnetic field B satisfying the estimates (1.32) in addition satisfies the zero flux condition

$$\Phi = \int_{\mathbb{R}^2} B(x)\mathrm{d}x = 0, \quad B(x) = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}, \tag{1.34}$$

then a magnetic vector-potential $A(x)$ satisfying the estimate (1.1) can be constructed in the same way as above. The necessity for the zero flux condition for dimension $d = 2$ is to ensure $U(x)$ in (1.33) does not depend on the choice of contour between x_0 and x. Let $B : \mathbb{R}^2 \to \mathbb{R}$ be given which satisfies the estimates (1.32) but the flux $\Phi \neq 0$. Then any magnetic vector potential $A(x)$ for this field will necessarily fail to be short-range (i.e. (1.1) fails) but one can construct $A(x)$ with the behaviour $A(x) \sim |x|^{-1}$ as $|x| \to \infty$ (here $a \sim b$ means $\frac{a}{b} \to 1$). In this case scattering theory for the operators H_0 and H can still be constructed, but the difference $S(k) - I$ will *not* be compact, see [32] for a detailed discussion and a description of the essential spectrum of $S(k)$. A particularly well known example of this is the Aharonov-Bohm effect [2]. Thus, in this case the eigenvalue counting measure (1.5) cannot even be defined and the question of the spectral asymptotics of the scattering matrix cannot be approached in the same way as above.

1.7. **The method of proof.** The proofs of all the results stated above follow a similar general structure, which we shall outline here. The specifics of each proof we shall introduce and explain further in the relevant chapters later in the text.

To begin, we note that it is known (see [1]) that off the diagonal $\omega = \omega'$, the integral kernel $s(\omega, \omega'; k)$ of the scattering matrix is a C^∞ smooth function of $\omega, \omega' \in \mathbb{S}^{d-1}$ and it tends to zero faster than any power of k^{-1} as $k \to \infty$. Thus, it suffices to describe the structure of $s(\omega, \omega'; k)$ in a neighbourhood of the diagonal $\omega = \omega'$.

Using this fact, we reduce the scattering matrix to a semiclassical pseudodifferential operator (ΨDO) $\mathrm{Op}_{k^{-1}}[\sigma]$ on the sphere in the form (2.1), with semiclassical parameter $h = k^{-1}$. The method used to do this is different for each case considered. For Theorem 1.8 (of which Theorem 1.5 is a special case), we use the Born approximation to the scattering matrix. For Theorem 1.9, we use an approximation obtained by Yafaev in [31], which is derived from the eikonal approximation to solutions of the Schrödinger equation. We remark here that when $A \equiv 0$, the first term of the eikonal approximation becomes the Born approximation.

Using the ΨDO calculus (given in Section 2), we compute $\mathrm{Tr}(\mathrm{Op}_{k^{-1}}[\sigma])^\ell$ (and similar objects involving adjoints) for natural numbers ℓ. This is extended to calculate the asymptotics as $k \to \infty$ of $\mathrm{Tr}(S(k) - I)^{\ell_1}(S(k)^* - I)^{\ell_2}$ for natural numbers ℓ_1, ℓ_2. Then the results follow in each case by an application of the Weierstrass approximation theorem.

We note here that we provide an alternate proof of Theorem 1.5 using the eikonal approximation. Note that when the magnetic vector-potential A in Theorem 1.9 is identically zero, the Schrödinger operators (and hence the associated scattering matrices) in Theorems 1.5 and 1.9 are identical. Using this method, we are able to work directly with the scattering matrix as opposed to the Born approximation. By rescaling by the parameter k, as in Theorem 1.5, we are able to obtain the same spectral asymptotics as provided by the result (1.19).

We note here that the result Theorem 1.5 appeared in [7], and Theorem 1.9 in [8].

1.8. **Overview of related results.** The limiting measures μ^e and μ^m (defined in (1.17) and (1.29) respectively) of the corresponding eigenvalue counting measures arise via integration over straight lines, that is over the trajectories of the free dynamics. Let us now mention some other works with similar characteristics. The classical result [28] by A. Weinstein concerns the Schrödinger operator for the Laplace-Beltrami operator on a compact Riemannian manifold with a smooth potential V. In the unperturbed case where the Riemannian manifold is chosen to be the unit sphere $\mathbb{S}^{d-1}$, the spectrum of the Laplace-Beltrami operator consists solely of eigenvalues, and the p^{th} eigenvalue is of the form $p(p + d - 1)$ with multiplicities growing as a polynomial of order $d - 1$ in p. Due to the presence of the potential V, the eigenvalues of the 'full' operator form clusters. Weinstein determined the asymptotic distribution of the eigenvalues in these clusters which is expressed as an integral over trajectories of the free dynamics of the bicharacteristic flow. The proof relied on the use of pseudodifferential operators on the Riemannian manifold.

More recent work in this area includes [26], [27] and [16]. In particular, the work [16] by A. Pushnitski, G. Raikov and C. Villegas-Blas describes the asymptotic density of eigenvalue clusters for the perturbed Landau Hamiltonian with a perturbation V satisfying the same conditions as the perturbation V of Theorem 1.5. The limiting density obtained in this work is expressed in terms of the X-ray transform of V (1.14). Their proof in part relies on the consideration of the asymptotics of a related pseudodifferential operator, similar in structure to that of the Born approximation. The proof of Theorem 1.5 is inspired by the method of proof used in [16].

1.9. **List of common notation.** Here we summarise the common notation that we use.

$C_0^\infty(\mathbb{R}^d)$ – the set of smooth scalar valued functions with a compact support in $\mathbb{R}^d$.

$C(\mathbb{R}^d)$ – the set of continuous scalar valued functions on $\mathbb{R}^d$.

$L^p(\mathbb{R}^d)$ – the set of measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that

$$\|f\|_{L^p}^p = \int_{\mathbb{R}^d} |f|^p \mathrm{d}x < +\infty.$$

$L^\infty(\mathbb{R}^d)$ – the set of measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that

$$\|f\|_{L^\infty} = \text{ess-sup}\,|f| < +\infty,$$

where ess-sup f denotes the essential supremum of f.

$\|\cdot\|_\ell$ – the usual Schatten ℓ-norm as described in Appendix A.

$\operatorname{Tr} A$ – the trace of a trace class operator A as described in Appendix A.

$\operatorname{Im} A$ – for a bounded linear operator A,

$$\operatorname{Im} A = \frac{A - A^*}{2i}.$$

$\operatorname{Re} A$ – for a bounded linear operator A,

$$\operatorname{Re} A = \frac{A + A^*}{2}.$$

$\langle x \rangle$ – for $x \in \mathbb{R}^d$, $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$.

2. Semiclassical pseudodifferential operators on the sphere

For every $\omega \in \mathbb{S}^{d-1}$, we identify the cotangent space $T^*_\omega \mathbb{S}^{d-1}$ with the plane $\Lambda_\omega = \{x \in \mathbb{R}^d : \langle x, \omega\rangle = 0\}$ in a standard way via the inner product in $\mathbb{R}^d$. For a symbol $\sigma \in C_0^\infty(T^*\mathbb{S}^{d-1})$ and a semiclassical parameter $h \in (0,1)$, the semiclassical ΨDO $\mathrm{Op}_h[\sigma]$ in $L^2(\mathbb{S}^{d-1})$ is defined via its integral kernel

$$\mathrm{Op}_h[\sigma](\omega,\omega') = (2\pi h)^{-d+1} \int_{\Lambda_\omega} e^{-i\langle \omega-\omega',\xi\rangle/h}\sigma(\omega,\xi)\mathrm{d}\xi, \tag{2.1}$$

where $\omega,\omega' \in \mathbb{S}^{d-1}$. This definition can be extended in a standard way to symbols σ satisfying

$$|\partial_\xi^\alpha \partial_\omega^\beta \sigma(\omega,\xi)| \le C_{\alpha\beta}\langle \xi\rangle^{m-|\alpha|}, \quad \omega \in \mathbb{S}^{d-1}, \quad \xi \in \Lambda_\omega, \tag{2.2}$$

for some $m \in \mathbb{R}$ and all multi-indices α,β. For all $m \le 0$, the operator $\mathrm{Op}_h[\sigma]$ is bounded (cf. [23]). We will only be interested in the case $m < 0$, and by the Calderon-Villancourt theorem (see e.g. [25]) combined with a scaling argument, we obtain the estimate

$$\sup_{0<h<1} \|\mathrm{Op}_h[\sigma]\| \le C(\sigma). \tag{2.3}$$

Remark 2.1. If σ satisfies (2.2) with $m < -(d-1)$, then (see e.g. [23, 10]) $\mathrm{Op}_h[\sigma]$ is trace class and its trace can be computed by integrating the kernel (2.1) over the diagonal:

$$\mathrm{Tr}(\mathrm{Op}_h[\sigma]) = (2\pi h)^{-d+1}\int_{\mathbb{S}^{d-1}}\int_{\Lambda_\omega} \sigma(\omega,\xi)\mathrm{d}\xi\mathrm{d}\omega. \tag{2.4}$$

Further, there exist constants C and N such that we have the following estimate on the trace class norm

$$\|\,\mathrm{Op}_h[\sigma]\|_1 \le C \sum_{|\alpha+\beta|\le N} \int |\partial_x^\alpha \partial_\omega^\beta \sigma(\omega,\xi)|\mathrm{d}\xi\mathrm{d}\omega. \tag{2.5}$$

Then it follows by using abstract interpolation (see e.g.[18] Section IX.4) on the estimates (2.3) and (2.5) that if σ satisfies (2.2) with $m < -\frac{d-1}{p}$, then $\mathrm{Op}_h[\sigma]$ belongs to the Schatten class S_p and

$$\|\mathrm{Op}_h[\sigma]\|_p \le Ch^{-(d-1)/p}, \quad h \in (0,1). \tag{2.6}$$

We will also be interested in symbols which depend on h. For $m < 0$, let $S^m(T^*\mathbb{S}^{d-1})$ be the class of C^∞-smooth symbols $\sigma = \sigma(\omega,\xi,h)$, $h \in (0,1)$, satisfying the estimate (2.2) uniformly in $h \in (0,1)$ for all multi-indices α,β. We will need a standard statement about the leading term spectral asymptotics of a semiclassical ΨDO. We provide two propositions here without proof - their proofs may be found in Appendix C. The first proposition we provide is simply a variant of the composition formula for pseudodifferential operators (see e.g.[23] Theorem 23.6).

Proposition 2.2. *Let $a \in S^m(T^*\mathbb{S}^{d-1})$, $b \in S^k(T^*\mathbb{S}^{d-1})$, $m < 0$, $k < 0$, and let $p \geq 1$ be such that $m + k < -\frac{d-1}{p}$. Then*

$$\| \operatorname{Op}_h[a] \operatorname{Op}_h[b] - \operatorname{Op}_h[ab] \|_p = O(h^{-\frac{d-1}{p}+1}), \quad h \to +0.$$

Proposition 2.3. *Let $a \in S^m(T^*\mathbb{S}^{d-1})$, $m < 0$, and let $p \geq 1$ be such that $m < -\frac{d-1}{p}$. Then*

$$\|(\operatorname{Op}_h[a])^* - \operatorname{Op}_h[\bar{a}]\|_p = O(h^{-\frac{d-1}{p}+1}), \quad h \to +0.$$

Proposition 2.4. *Let $\sigma \in S^m(T^*\mathbb{S}^{d-1})$, $m < 0$, and let ℓ_1, ℓ_2, be two non-negative integers such that $m < -\frac{d-1}{\ell_1+\ell_2}$. Then $(\operatorname{Op}_h[\sigma])^{\ell_1}\big((\operatorname{Op}_h[\sigma])^*\big)^{\ell_2}$ belongs to the trace class and*

(2.7)

$$\begin{aligned} &\operatorname{Tr}\big((\operatorname{Op}_h[\sigma])^{\ell_1}\big((\operatorname{Op}_h[\sigma])^*\big)^{\ell_2}\big) = \\ &\qquad (2\pi h)^{-d+1}\int_{\mathbb{S}^{d-1}}\int_{\Lambda_\omega} \sigma_0(\omega,\xi)^{\ell_1}\overline{\sigma_0(\omega,\xi)^{\ell_2}}\,\mathrm{d}\xi\mathrm{d}\omega + O(h^{-d+2}), \quad h \to +0. \end{aligned}$$

Proof. Let $\ell = \ell_1 + \ell_2$. By Proposition 2.3,

$$\|(\operatorname{Op}_h[\sigma])^* - \operatorname{Op}_h[\bar{\sigma}]\|_\ell = O(h^{-\frac{d-1}{\ell}+1}), \quad h \to +0.$$

Next, by Proposition 2.2,

$$\|(\operatorname{Op}_h[\sigma])^2 - \operatorname{Op}_h[\bar{\sigma}^2]\|_{\ell/2} = O(h^{-\frac{2(d-1)}{\ell}+1}), \quad h \to +0,$$

and by repeating this process we obtain

$$\|(\operatorname{Op}_h[\sigma])^j - \operatorname{Op}_h[\bar{\sigma}^j]\|_{\ell/j} = O(h^{-\frac{j(d-1)}{\ell}+1}), \quad h \to +0,$$

for $j = 1, 2, \dots, \ell_1$. In the same way, we get a similar result for the adjoint operator:

$$\|((\operatorname{Op}_h[\sigma])^*)^j - \operatorname{Op}_h[\bar{\sigma}^j]\|_{\ell/j} = O(h^{-\frac{j(d-1)}{\ell}+1}), \quad h \to +0.$$

Thus, using Proposition 2.2 again,

$$\|(\operatorname{Op}_h[\sigma])^{\ell_1}(\operatorname{Op}_h[\sigma])^*)^{\ell_2} - \operatorname{Op}_h[\sigma^{\ell_1}\bar{\sigma}^{\ell_2}]\|_1 = O(h^{-(d-1)+1}), \quad h \to +0.$$

It now remains to apply the formula (2.4) for the trace of a pseudodifferential operator. □

In our construction, the ΨDO will be defined in terms of their amplitudes rather than their symbols. Thus, we need a statement which is standard in the ΨDO theory (see e.g. [23]).

Proposition 2.5. *Let $m < 0$, and let $b = b(\omega, \omega', \xi, h)$ be a smooth function of the variables $(\omega, \xi) \in T^*\mathbb{S}^{d-1}$, $\omega' \in \mathbb{S}^{d-1}$ and $h \in (0, 1)$. Assume that b satisfies the estimates*

$$|\partial_\xi^\alpha \partial_\omega^\beta \partial_{\omega'}^\gamma b(\omega, \omega', \xi, h)| \leq C_{\alpha\beta\gamma}\langle \xi \rangle^{m-|\alpha|} \tag{2.8}$$

for all multi-indices α, β, γ *uniformly in* $h \in (0,1)$. *Then for any* $N > 0$, *the operator with the integral kernel*

$$(2\pi h)^{-d+1} \int_{\Lambda_\omega} e^{-i\langle \omega-\omega', \xi\rangle/h} b(\omega, \omega', \xi, h) \mathrm{d}\xi \tag{2.9}$$

can be represented as $\mathrm{Op}_h[\sigma] + R_N(h)$, *where the following conditions are met:*

(i) *The symbol* σ *can be written as* $\sigma = \sigma_0 + h\sigma_1$ *with* $\sigma_0, \sigma_1 \in S^m(T^*\mathbb{S}^{d-1})$ *and*

$$\sigma_0(\omega, \xi, h) = b(\omega, \omega, \xi, h).$$

(ii) *The operator* $R_N(h)$ *has the integral kernel in* $C^N(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$ *with* C^N*-norm satisfying* $O(h^N)$ *as* $h \to 0$.

3. Proof of Theorem 1.8

We shall here provide the proof of Theorem 1.8 (and as a direct consequence Theorem 1.5). We begin by discussing the limiting absorption principle for the Schrödinger operator $H = -\Delta + \alpha V$, where the coupling constant $\alpha > 0$. We shall then provide the stationary representation of the scattering matrix together with some important estimates.

The proof begins by assuming that $V \in C_0^\infty(\mathbb{R}^d)$. The scattering matrix is then expressed (via the Born approximation) as a semiclassical ΨDO on the sphere. We obtain the leading spectral asymptotics as $k \to \infty$ for the scattering matrix in this case. The result is then extended to cover all continuous potentials satisfying the short-range condition (1.2), and Theorem 1.8 is proved by an application of the Weierstrass approximation theorem.

3.1. **The limiting absorption principle and its consequences.** We shall here state the limiting absorption principle for the Schrödinger operator $H = -\Delta + \alpha V$ where $\alpha > 0$. We then provide the stationary representation of the scattering matrix, and derive some important estimates. Let us begin by defining some notation. Let B_ρ be the normed linear space of all continuous potentials V satisfying the short-range condition (1.2) with the norm

$$\|V\|_{B_\rho} = \sup_{x \in \mathbb{R}^d} |V(x)| \langle x \rangle^\rho. \tag{3.1}$$

By $T(z;\alpha)$ we denote the sandwiched resolvent

$$T(z;\alpha) = \langle x \rangle^{-\rho/2} (H - zI)^{-1} \langle x \rangle^{-\rho/2}, \quad \operatorname{Im} z > 0. \tag{3.2}$$

The statement of the limiting absorption principle is as follows (see e.g. [30] Proposition 7.1.4)

Proposition 3.1. *Let $H = -\Delta + \alpha V$, where V satisfies the short-range condition* (1.2). *Let $z \in \mathbb{C} \setminus [0, +\infty)$ with $|z| \geq c > 0$. If the coupling constant α satisfies $\alpha = o(|z|^{\frac{1}{2}})$ as $|z| \to \infty$, then*

$$\|T(z;\alpha)\| \leq C|z|^{-\frac{1}{2}}, \quad \operatorname{Im} z > 0. \tag{3.3}$$

In order to give the stationary representation of the scattering matrix $S(k)$ associated with the operators $H_0 = -\Delta$ and $H = -\Delta + \alpha V$ where α satisfies the condition (1.21) for $\delta \in [0, 1)$, we shall require some more notation. Let us factorize V as

$$V = \langle x \rangle^{-\rho/2} J \langle x \rangle^{-\rho/2} \tag{3.4}$$

where $J = J(\rho)$ is the bounded operator of multiplication by $\langle x\rangle^\rho V(x)$. Next, for $k > 0$ and $\rho > 1$ we define the operator $\Gamma_\rho(k) : L^2(\mathbb{R}^d) \to L^2(\mathbb{S}^{d-1})$ by

$$(\Gamma_\rho(k)u)(\omega) = \frac{1}{\sqrt{2}} k^{(d-2)/2}(2\pi)^{-d/2} \int_{\mathbb{R}^d} u(x)\langle x\rangle^{-\rho/2} e^{-ik\langle x,\omega\rangle} \mathrm{d}x, \quad \omega \in \mathbb{S}^{d-1}, \tag{3.5}$$

(that is, $\Gamma_\rho(k)$ is the composition of the operators Γ_0 in (1.11) and $\langle x\rangle^{-\rho/2}$ for a given value of $k > 0$). We claim that this operator is well defined. To see this, first consider a map $T_1 : L^2(\mathbb{R}^d) \to H^\rho(\mathbb{R}^d)$ given by

$$(T_1 u)(p) = \int_{R^d} u(x)\langle x\rangle^{-\rho/2} e^{-ip\cdot x} \mathrm{d}x, \quad \rho > 1.$$

Next consider the operator of restriction $T_2 : H^\rho(\mathbb{R}^d) \to H^\rho(\mathbb{S}^{d-1}_k)$ onto the sphere $\mathbb{S}^{d-1}_k$ of radius k. It then follows from the Sobolev trace theorem that the operator T_2 is a bounded operator in the space $L^2(\mathbb{S}^{d-1})$ since $\rho > 1$. The operator $\Gamma_\rho(k)$ is given by the composition of operators T_1 and T_2 and hence the above claim is justified. Then the scattering matrix may be expressed in the stationary representation by

$$S(k) - I = -2\pi i\alpha\Gamma_\rho(k)[J - \alpha JT(k^2 + i0; \alpha)J]\Gamma_\rho(k)^*, \quad k > 0, \tag{3.6}$$

where the limit $T(k^2 + i0; \alpha) = \lim_{\varepsilon\to 0} T(k^2 + i\varepsilon; \alpha)$ exists courtesy of Proposition 3.1. We next provide an estimate in the Schatten norm for the operator $\Gamma_\rho(k)$: this result is well known (see e.g. [30, Lemma 8.1.2]).

Lemma 3.2. *For $\rho > 1$ and any $\ell \geq 1$ such that $\ell > \frac{d-1}{\rho-1}$, the following estimate holds:*

$$\sup_{k\geq 1} k^{\frac{\ell+1-d}{\ell}} \|\Gamma_\rho(k)\|_{2\ell}^2 \leq C(\ell, \rho, d). \tag{3.7}$$

Proof. Let us first prove (3.7) for $\ell = 1$. From (3.5) and the formula (A.2) for the Hilbert-Schmidt norm, we see that

$$\|\Gamma_\rho(k)\|_2^2 = 2^{-1}k^{d-2}(2\pi)^{-d} \int_{\mathbb{R}^d} \langle x\rangle^{-\rho} \mathrm{d}x.$$

The above integral is finite exactly when $\rho > d$ and hence

$$\sup_{k\geq 1} k^{2-d} \|\Gamma_\rho(k)\|_2^2 \leq C(\ell, \rho, d), \quad \rho > d. \tag{3.8}$$

We now prove (3.7) for the operator norm i.e. when $\ell = \infty$. From the spectral representation of H_0, it follows that

$$\|\Gamma_\rho(k)f\|^2 = \pi^{-1}(\mathrm{Im}\, R_0(k^2 + i0)\langle x\rangle^{-\rho/2} f, \langle x\rangle^{-\rho/2} f), \quad f \in L^2(\mathbb{R}^d).$$

Hence it follows that

$$\|\Gamma_\rho(k)f\|^2 \leq C\|T_0(k^2 + i0)\| \, \|f\|^2 \tag{3.9}$$

where $T_0(z)$ is the sandwiched resolvent

$$T_0(z) = \langle x\rangle^{-\rho/2}(H_0 - zI)^{-1}\langle x\rangle^{-\rho/2}, \quad \mathrm{Im}\, z \neq 0.$$

Thus combining estimates (3.3) and (3.9) it follows that

$$\sup_{k\geq 1} k\|\Gamma_\rho(k)\|^2 \leq C. \tag{3.10}$$

By using abstract interpolation (see e.g. [18] Section IX.4) on the estimates (3.8) and (3.10), the estimate (3.7) follows for $\ell > \frac{d-1}{\rho-1}$ as required. ☐

For ease of notation, from here onwards we shall denote by $\Upsilon = \Upsilon(\alpha)$ the quantity

$$\Upsilon = k\alpha^{-1}, \quad \alpha = O(k^\delta), \quad \delta \in [0,1). \tag{3.11}$$

We now give a norm estimate on the difference $S(k) - I$ in (3.6).

Lemma 3.3. *For $\rho > 1$ and any $\ell \geq 1$ such that $\ell > \frac{d-1}{\rho-1}$, the following estimate holds:*

$$\sup_{k\geq 1} k^{\frac{1-d}{\ell}} \|\Upsilon(S(k) - I)\|_\ell \leq C(\ell,\rho,d)\|V\|_{B_\rho}. \tag{3.12}$$

Proof. From (3.6) and the usual Hölder-type inequality for the Schatten norm (see the Appendix A), we have that

$$\|S(k) - I\|_\ell \leq C\alpha\|\Gamma_\rho(k)\|^2_{2\ell}[\|J\| + \alpha\|J\|^2\|T(k^2 + i0;\alpha)\|]$$

Since $\|J\| = \|V\|_{B_\rho}$, we have that

$$k^{\frac{1-d}{\ell}}\|\Upsilon(S(k) - I)\|_\ell \leq C(\ell,\rho,d)\left(k^{\frac{\ell+1-d}{\ell}}\|\Gamma_\rho(k)\|^2_{2\ell}\right)\left[\|V\|_{B_\rho} + \|T(k^2+i0;\alpha)\|\|V\|^2_{B_\rho}\right]$$

and the result follows by applying (3.3) and (3.7) to the above. ☐

Recall that the spectral asymptotics for the scattering matrix when $\alpha = O(k^\delta)$, $\delta \in [0,1)$ as $k \to \infty$ are derived from the Born approximation, which is defined as

$$S_B(k) = I - 2\pi i\alpha\Gamma_\rho(k)J\Gamma_\rho(k)^*.$$

We shall in fact consider the imaginary part of the Born approximation, which is given by

$$\operatorname{Im} S_B(k) = -2\pi\alpha\Gamma_\rho(k)J\Gamma_\rho(k)^*. \tag{3.13}$$

Proposition 3.4. *Let $V \in B_\rho$ with $\rho > 1$ and let $\alpha = O(k^\delta)$ for $\delta \in [0,1)$ as $k \to \infty$. Then for any $\ell \geq 1$ such that $\ell > \frac{d-1}{\rho-1}$*

$$\sup_{k\geq 1} k^{\frac{1-d}{\ell}}\|\Upsilon \operatorname{Im} S_B(k)\|_\ell \leq C(\ell,\rho,d)\|V\|_{B_\rho}, \tag{3.14}$$

$$\sup_{k\geq 1} k^{\frac{1-d}{\ell}}\|\Upsilon^2 \operatorname{Im}(S_B(k) - S(k))\|_\ell \leq C(\ell,\rho,d,V), \tag{3.15}$$

$$\sup_{k\geq 1} k^{\frac{1-d}{\ell}}\|\Upsilon \operatorname{Im} S(k)\|_\ell \leq C(\ell,\rho,d,V). \tag{3.16}$$

Proof. Let us prove each estimate in turn. First, by (3.13) and the usual Hölder-type inequality for the Schatten norm (see the Appendix A),

$$\|\mathrm{Im}\, S_B(k)\|_\ell \leq C\alpha \|\Gamma_\rho(k)\|_{2\ell}^2 \|J\|.$$

Note that

$$\|J\| = \|V\|_{B_\rho}. \tag{3.17}$$

Then the result (3.14) follows from (3.7) and by recalling the value of Υ.
Next, it is easily shown that

$$\mathrm{Im}\,(S_B(k) - S(k)) = -2\pi\alpha^2 (\Gamma_\rho(k)J) \mathrm{Re}\, T(k^2 + i0; \alpha)(\Gamma_\rho(k)J)^*.$$

Thus

$$\|\mathrm{Im}\,(S_B(k) - S(k))\|_\ell \leq C\alpha^2 \|\Gamma_\rho(k)\|_{2\ell}^2 \|J\| \|\mathrm{Re}\, T(k^2 + i0; \alpha)\|,$$

where the result (3.15) follows from (3.3), (3.7) and by recalling the value of Υ.
Finally, the estimate (3.16) follows by combining (3.14) and (3.15). $\square$

Lemma 3.5. *Let $V \in B_\rho$ with $\rho > 1$ and let $\alpha = O(k^\delta)$ for $\delta \in [0, 1)$ as $k \to \infty$. Then for any integer $\ell \geq 1$ satisfying $\ell > \frac{d-1}{\rho-1}$, one has*

$$|\operatorname{Tr}(\Upsilon \mathrm{Im}\, S(k))^\ell - \operatorname{Tr}(\Upsilon \mathrm{Im}\, S_B(k))^\ell| = O(k^{d-2+\delta}), \quad \delta \in [0, 1), \quad k \to \infty.$$

Proof. We begin by stating the following factorization for linear operators:

$$A^\ell - B^\ell = \sum_{j=0}^{\ell-1} A^j (A - B) B^{\ell-1-j}.$$

Next since $|\operatorname{Tr} A| \leq \|A\|_1$,

$$|\operatorname{Tr}(A^\ell - B^\ell)| \leq \sum_{j=0}^{\ell-1} \|A^j (A - B) B^{\ell-1-j}\|_1. \tag{3.18}$$

Note that by the Hölder inequality for trace ideals (A.1), for ℓ operators $A_1, \ldots, A_\ell \in S_\ell$, we have the estimate

$$\|A_1 A_2 \ldots A_\ell\|_1 \leq \|A_1\|_\ell \|A_2 \ldots A_\ell\|_{\frac{\ell}{\ell-1}}$$

where in (A.1) we have chosen $\ell_1 = \ell$, $\ell_2 = \frac{\ell}{\ell-1}$. By using this repeatedly we obtain

$$\|A_1 A_2 \ldots A_\ell\|_1 \leq \prod_{i=1}^{\ell} \|A_i\|_\ell, \quad A_1, \ldots, A_\ell \in S_\ell.$$

It follows from the above estimate that

$$\|A^j (A - B) B^{\ell-1-j}\|_1 \leq \|A\|_\ell^j \|A - B\|_\ell \|B\|_\ell^{\ell-1-j}. \tag{3.19}$$

Combining (3.18) and (3.19) yields

$$|\operatorname{Tr}(A^\ell - B^\ell)| \le \ell \|A-B\|_\ell \max\{\|A\|_\ell^{\ell-1}, \|B\|_\ell^{\ell-1}\}, \quad A,B \in S_\ell. \tag{3.20}$$

Thus, it suffices to prove the relation

$$\begin{aligned}&\|\Upsilon \mathrm{Im}\,(S(k)-S_B(k))\|_\ell \times\\ &\times \max\{\|\Upsilon \mathrm{Im}\,S(k)\|_\ell^{\ell-1}, \|\Upsilon \mathrm{Im}\,S_B(k)\|_\ell^{\ell-1}\} = O(k^{d-2+\delta}), \quad k\to\infty.\end{aligned}$$

The latter relation follows by combining (3.14)–(3.16) and recalling $\Upsilon^{-1} = O(k^{\delta-1})$ as $k\to\infty$. □

3.2. **The Born approximation.** We begin by assuming that $V \in C_0^\infty(\mathbb{R}^d)$ and expressing the imaginary part of the Born approximaton $\mathrm{Im}\,S_B(k)$ as a semiclassical ΨDO on the sphere with semiclassical parameter $h = k^{-1}$ and symbol $X(\omega,\xi)$. We then prove, using Proposition 2.4 that for any $\ell \in \mathbb{N}$,

$$\lim_{k\to\infty} k^{1-d} \operatorname{Tr}(\Upsilon \mathrm{Im}\,S_B(k))^\ell = \int_{-\infty}^{\infty} t^\ell \mathrm{d}\mu^e(t), \quad \alpha = O(k^\delta), \quad \delta\in[0,1), \tag{3.21}$$

where the measure μ^e is defined in (1.6). By applying an approximation argument we extend the result (3.21) to all $V \in B_\rho$ with $\rho > 1$ and any $\ell \ge 1$ satisfying $\ell > \frac{d-1}{\rho-1}$.

Proposition 3.6. *Let $V \in C_0^\infty(\mathbb{R}^d)$ and let $\alpha = O(k^\delta)$ for $\delta\in[0,1)$ as $k\to\infty$. Then the Born approximation may be expressed as*

$$\Upsilon \mathrm{Im}\,S_B(k) = \mathrm{Op}_{k^{-1}}[\sigma] + R(k) \tag{3.22}$$

where the following conditions are met:
1) The symbol σ can be represented as

$$\sigma = \sigma_0 + k^{-1}\sigma_1 \tag{3.23}$$

where $\sigma_0, \sigma_1 \in C_0^\infty(T^*\mathbb{S}^{d-1})$ are uniformly bounded in k together with all derivatives. Here $T^*\mathbb{S}^{d-1}$ denotes the cotangent bundle of the sphere, as discussed in Section 2. Further, we have the relation

$$\sigma_0(\omega,\xi) = X(\omega,\xi) \tag{3.24}$$

with the function X as in (1.14).
2) The operator $R(k)$ has integral kernel which belongs to $C^\infty(\mathbb{S}^{d-1}\times\mathbb{S}^{d-1})$ and its $C(\mathbb{S}^{d-1}\times\mathbb{S}^{d-1})$-norm is $O(k^{-\infty})$ as $k\to\infty$.

Before giving the proof, let us note that we only require the $C(\mathbb{S}^{d-1}\times\mathbb{S}^{d-1})$-norm for the operator $R(k)$ and not any of its derivatives since we shall only be taking the trace of $R(k)$ and not requiring any norm estimates.

Proof. 1) For ease of notation, we shall write $Q(k) = \Upsilon \mathrm{Im}\, S_B(k)$. By (3.5) and (3.13), $Q(k)$ is the integral operator in $L^2(\mathbb{S}^{d-1})$ with the integral kernel

$$Q(k)(\omega,\omega') = -2^{-1}k^{d-1}(2\pi)^{1-d}\int_{\mathbb{R}^d} e^{-ik\langle\omega-\omega',x\rangle}V(x)\mathrm{d}x, \quad \omega,\omega'\in\mathbb{S}^{d-1}. \tag{3.25}$$

Let us introduce a function $\chi_0 \in C^\infty(\mathbb{S}^{d-1}\times\mathbb{S}^{d-1})$ such that $\chi_0(\omega,\omega')=1$ in a neighbourhood of the diagonal $\omega=\omega'$ and whose support is the set

$$\big\{(\omega,\omega')\in\mathbb{S}^{d-1}\times\mathbb{S}^{d-1} : \langle\omega,\omega'\rangle\geq\gamma\big\}$$

for some $\gamma\in(\frac{\sqrt{2}}{2},1)$. Denote $\chi_1 = 1-\chi_0$ and let

$$Q(k) = Q_0(k) + Q_1(k)$$

where $Q_j(k)$ is the operator with integral kernel $\chi_j(\omega,\omega')Q(k)(\omega,\omega')$. By the fast decay of the Fourier transform of V, and by the fact that $|\omega-\omega'|$ is separated away from zero on the support of χ_1, we see that

$$\sup_{\omega,\omega'}|Q_1(k)(\omega,\omega')| = O(k^{-\infty}), \quad k\to\infty.$$

Thus, it suffices to concern ourselves with $Q_0(k)$.

2) Let us change integration variables in $Q_0(k)$. For fixed $\omega,\omega'\in\mathbb{S}^{d-1}$ such that $\omega+\omega'\neq 0$, we define the vector

$$\nu = \nu(\omega,\omega') = \frac{\omega+\omega'}{|\omega+\omega'|}\in\mathbb{S}^{d-1}. \tag{3.26}$$

We set

$$x = t\nu+\xi, \quad \xi\in\Lambda_\omega, \quad t = \frac{\langle x,\omega\rangle}{\langle\nu,\omega\rangle}. \tag{3.27}$$

By the orthogonality relation $(\omega-\omega')\perp\nu$, one has

$$\langle\omega-\omega',x\rangle = \langle\omega-\omega',\xi\rangle.$$

Using this change of variables in the definition of $Q_0(k)$ yields

$$Q_0(k)(\omega,\omega') = k^{d-1}(2\pi)^{1-d}\int_{\Lambda_\omega} e^{-ik\langle\omega-\omega',\xi\rangle}b(\omega,\omega',\xi)\mathrm{d}\xi, \quad \omega,\omega'\in\mathbb{S}^{d-1},$$

where

$$b(\omega,\omega',\xi) = -2^{-1}\chi_0(\omega,\omega')J(\omega,\omega')\int_{-\infty}^{\infty}V(t\nu+\xi)\mathrm{d}t$$

and $J(\omega,\omega')$ denotes the Jacobian obtained from the change of variables (3.27). It is easy to see that $J(\omega,\omega')$ is a smooth function of $\omega,\omega'\in\operatorname{supp}\chi_0$. Let us calculate $J(\omega,\omega')$ on the diagonal $\omega=\omega'$. In this case, the change of variables (3.27) becomes

$$x = t\omega+\xi, \quad \omega\in\mathbb{S}^{d-1}, \quad \xi\in\Lambda_\omega, \quad t=\langle x,\omega\rangle.$$

This is simply an orthogonal change of variables, and it is straightforward to see that

$$J(\omega,\omega)=1. \tag{3.28}$$

3) In order to complete the proof, by Proposition 2.5 it suffices to check the estimate

$$|\partial_x^\alpha\partial_\omega^\beta\partial_{\omega'}^\gamma b(\omega,\omega',\xi)|\le C_{\alpha\beta\gamma}\langle\xi\rangle^{m-|\alpha|} \tag{3.29}$$

for some $m<0$ and all multi-indices α,β,γ uniformly over $k\ge 1$, and to check the identity

$$b(\omega,\omega,\xi)=X(\omega,\xi). \tag{3.30}$$

The estimate (3.29) is almost immediate from the fact that $V\in C_0^\infty(\mathbb{R}^d)$ and since the amplitude b is smooth in all variables. The identity (3.30) is also apparent from recalling the definition of the function X from (1.14), the equation (3.28) and noting that $\chi_0(\omega,\omega)=1$ and $\nu(\omega,\omega)=\omega$. □

Lemma 3.7. *Let $V\in C_0^\infty(\mathbb{R}^d)$ and let $\alpha=O(k^\delta)$ for $\delta\in[0,1)$ as $k\to\infty$. Then for any natural number ℓ, we have*

$$\lim_{k\to\infty}k^{1-d}\operatorname{Tr}(\Upsilon\operatorname{Im}S_B(k))^\ell=\int_{-\infty}^{\infty}t^\ell\mathrm{d}\mu^e(t) \tag{3.31}$$

where the measure μ^e is defined in (1.6).

Proof. First let $\ell=1$. From (3.22), we have

$$\operatorname{Tr}\Upsilon\operatorname{Im}S_B(k)=\operatorname{Tr}\operatorname{Op}_{k^{-1}}[\sigma]+\operatorname{Tr}R(k). \tag{3.32}$$

Since the operator $R(k)$ has an integral kernel belonging to the class $C^\infty(\mathbb{S}^{d-1}\times\mathbb{S}^{d-1})$ with corresponding $C(\mathbb{S}^{d-1}\times\mathbb{S}^{d-1})$-norm equal to $O(k^{-\infty})$ as $k\to\infty$, it follows that

$$\lim_{k\to\infty}k^{1-d}\operatorname{Tr}R(k)=0. \tag{3.33}$$

Consequently, by applying (2.7) to (3.32) and recalling the equation (3.23) for σ,

$$\lim_{k\to\infty}k^{1-d}\operatorname{Tr}(\Upsilon\operatorname{Im}S_B(k))=(2\pi)^{1-d}\int_{\mathbb{S}^{d-1}}\int_{\Lambda_\omega}X(\omega,\xi)\mathrm{d}\xi\mathrm{d}\omega.$$

The result (3.31) follows in the case $\ell=1$ from the above and the definition (1.6) of the measure μ^e.

Now suppose that $\ell\ge 2$. Note that by (2.7),

$$\lim_{k\to\infty}k^{1-d}\operatorname{Tr}(\operatorname{Op}_{k^{-1}}[\sigma])^\ell=(2\pi)^{1-d}\int_{\mathbb{S}^{d-1}}\int_{\Lambda_\omega}[X(\omega,\xi)]^\ell\mathrm{d}\xi\mathrm{d}\omega.$$

Thus in order to prove (3.31), recalling (3.22) it suffices to prove that

$$\lim_{k\to\infty}k^{1-d}|\operatorname{Tr}(\Upsilon\operatorname{Im}S_B(k))^\ell-\operatorname{Tr}(\operatorname{Op}_{k^{-1}}[\sigma])^\ell|=0. \tag{3.34}$$

Applying the binomial theorem and the statement

$$|\operatorname{Tr}(AB)| \leq \|A\|_2 \|B\|_2, \quad A, B \in S_2, \tag{3.35}$$

to (3.34) yields

$$|\operatorname{Tr}(\Upsilon \operatorname{Im} S_B(k))^\ell - \operatorname{Tr}(\operatorname{Op}_{k^{-1}}[\sigma])^\ell| \leq C(\ell) \max_{1 \leq n \leq \ell} \|R(k)\|_2^n \|\operatorname{Op}_{k^{-1}}[\sigma]\|_2^{\ell - n}. \tag{3.36}$$

We easily obtain by direct calculation that

$$\|R(k)\|_2 = O(k^{-\infty}), \quad k \to \infty. \tag{3.37}$$

The result (3.34) follows from (3.36) together with the estimates (2.6) and (3.37). □

Lemma 3.8. *Let $V \in B_\rho$ with $\rho > 1$ and let $\alpha = O(k^\delta)$ for $\delta \in [0, 1)$ as $k \to \infty$. Then for any integer $\ell \geq 1$ satisfying $\ell > \frac{d-1}{\rho-1}$, we have*

$$\lim_{k \to \infty} k^{1-d} \operatorname{Tr}(\Upsilon \operatorname{Im} S_B(k))^\ell = \int_{-\infty}^{\infty} t^\ell \mathrm{d}\mu^e(t) \tag{3.38}$$

where the measure μ^e is defined in (1.6).

Proof. Let B_ρ^0 denote the closure of $C_0^\infty(\mathbb{R}^d)$ in B_ρ. For any $\ell > \frac{d-1}{\rho-1}$, set

$$\begin{aligned} g_\ell(V) &= \int_{-\infty}^{\infty} t^\ell \mathrm{d}\mu^e(t), \\ g_\ell^+(V) &= \limsup_{k \to \infty} k^{1-d} \operatorname{Tr}(\Upsilon \operatorname{Im} S_B(k))^\ell, \\ g_\ell^-(V) &= \liminf_{k \to \infty} k^{1-d} \operatorname{Tr}(\Upsilon \operatorname{Im} S_B(k))^\ell. \end{aligned}$$

Let us first show that $g_\ell, g_\ell^\pm$ are continuous functionals on B_ρ. We begin by showing $g_\ell(V)$ is continuous. Let $V_1, V_2 \in B_\rho$. Denote by $X_j(\omega, \xi)$ the function $X(\omega, \xi)$ in (1.14) with V replaced by V_j, $j = 1, 2$. By definition,

$$g_\ell(V_1) - g_\ell(V_2) = (2\pi)^{1-d} \int_{\mathbb{S}^{d-1}} \int_{\Lambda_\omega} \left([X_1(\omega, \xi)]^\ell - [X_2(\omega, \xi)]^\ell\right) \mathrm{d}\xi \mathrm{d}\omega. \tag{3.39}$$

We write

$$[X_1(\omega, \xi)]^\ell - [X_2(\omega, \xi)]^\ell = (X_1 - X_2)(\omega, \xi) \sum_{p=0}^{\ell-1} (X_1(\omega, \xi))^p (X_2(\omega, \xi))^{\ell-1-p}. \tag{3.40}$$

We here note the estimate

$$|X_j(\omega, \xi)| \leq C \|V_j\|_{B_\rho} \langle \xi \rangle^{1-\rho}, \quad j = 1, 2, \tag{3.41}$$

and further since B_ρ is a linear space,

$$|(X_1 - X_2)(\omega, \xi)| \leq C \|V_1 - V_2\|_{B_\rho} \langle \xi \rangle^{1-\rho}. \tag{3.42}$$

By inserting (3.40) into (3.39) and applying the estimates (3.41), (3.42),

$$|g_\ell(V_1) - g_\ell(V_2)| \le C(d,\ell)\|V_1 - V_2\|_{B_\rho}\times$$
$$\times \max\left\{\|V_1\|_{B_\rho}^{\ell-1}, \|V_2\|_{B_\rho}^{\ell-1}\right\} \int_{\Lambda_\omega} \langle\xi\rangle^{-\ell(\rho-1)} \mathrm{d}\xi.$$

We note that the integral in the above estimate is bounded above since $\ell(\rho-1) > d-1$, which suffices to prove that g_ℓ is a continuous functional on B_ρ.

Next let us show that $g_\ell^+(V)$ is continuous on B_ρ. Denote by $S_B^j(k)$ the corresponding Born approximation for the potential V_j, $j = 1,2$. Then we may write

$$(\Upsilon \mathrm{Im}\, S_B^1(k))^\ell - (\Upsilon \mathrm{Im}\, S_B^2(k))^\ell =$$
$$= \sum_{j=0}^{\ell-1} \left[\Upsilon \mathrm{Im}\, S_B^1(k)\right]^j \left[\Upsilon\left(\mathrm{Im}\,(S_B^1(k) - S_B^2(k))\right)\right] \left[\Upsilon \mathrm{Im}\, S_B^2(k)\right]^{\ell-1-j}$$

Applying the formula (3.20) to the above, we have

$$|\operatorname{Tr}(\Upsilon \mathrm{Im}\, S_B^1(k))^\ell - \operatorname{Tr}(\Upsilon \mathrm{Im}\, S_B^2(k))^\ell| \le$$
$$\le \ell\|\Upsilon \mathrm{Im}\,(S_B^1(k) - S_B^2(k))\|_\ell \times \max\{\|\Upsilon \mathrm{Im}\, S_B^1(k)\|_\ell^{\ell-1}, \|\Upsilon \mathrm{Im}\, S_B^2(k)\|_\ell^{\ell-1}\}.$$

It follows from the above that

$$\begin{aligned}|g_\ell^+(V_1) - g_\ell^+(V_2)| \le C \limsup_{k\to\infty} k^{1-d}\|\Upsilon \mathrm{Im}\,(S_B^1(k) - S_B^2(k))\|_\ell\times \\ \times \max\{\|\Upsilon \mathrm{Im}\, S_B^1(k)\|_\ell^{\ell-1}, \|\Upsilon \mathrm{Im}\, S_B^2(k)\|_\ell^{\ell-1}\}.\end{aligned} \tag{3.43}$$

Using the estimate (3.14), by the linearity of $\mathrm{Im}\, S_B(k)$ in V it follows from (3.43) that

$$|g_\ell^+(V_1) - g_\ell^+(V_2)| \le C(\ell,\rho,d)\|V_1 - V_2\|_{B_\rho} \max\{\|V_1\|_{B_\rho}^{\ell-1}, \|V_2\|_{B_\rho}^{\ell-1}\},$$

which suffices to demonstrate the continuity of $g_\ell^+(V)$ on B_ρ. The continuity of g_ℓ^- follows in a similar manner.

Next, note that by Lemma 3.7 for any $V \in C_0^\infty(\mathbb{R}^d)$,

$$g_\ell^-(V) = g_\ell(V) = g_\ell^+(V); \tag{3.44}$$

we would like to extend this to all $V \in B_\rho$. By choosing some $V \in B_\rho^0$, there exists a sequence $\{V_n\}_{n=1}^\infty \subset C_0^\infty(\mathbb{R}^d)$ such that

$$\lim_{n\to\infty} \|V_n - V\|_{B_\rho} = 0.$$

By the continuity of the functionals $g_\ell, g_\ell^\pm$ and the statement (3.44) for $V \in C_0^\infty(\mathbb{R}^d)$, it follows that (3.44) holds for all $V \in B_\rho^0$. Finally, for any $V \in B_\rho$ and a given $\ell > \frac{d-1}{\rho-1}$, choose ρ_1 such that $1 < \rho_1 < \rho$ with $\ell > \frac{d-1}{\rho_1-1}$. Then $B_\rho \subset B_{\rho_1}^0$ and the previous argument proves (3.44) for all $V \in B_{\rho_1}^0$ which suffices. $\square$

3.3. **From the Born approximation to the full scattering matrix.**

Lemma 3.9. *Let $V \in X_\rho$ with $\rho > 1$ and let $\alpha = O(k^\delta)$ for $\delta \in [0,1)$ as $k \to \infty$. Then for any integer $\ell \geq 1$ satisfying $\ell > \frac{d-1}{\rho-1}$,*

$$\lim_{k\to\infty} k^{1-d} \int_{-\infty}^{\infty} t^\ell \mathrm{d}\tilde{\mu}_k(t) = \int_{-\infty}^{\infty} t^\ell \mathrm{d}\mu^e(t).$$

Proof. By Lemmas 3.5 and 3.8, it suffices to prove

$$\lim_{k\to\infty} k^{1-d} \left| \int_{-\infty}^{\infty} t^\ell \mathrm{d}\tilde{\mu}_k(t) - \mathrm{Tr}(\Upsilon \mathrm{Im}\, S(k))^\ell \right| = 0. \tag{3.45}$$

Recalling the definition (1.6) of the measure $\tilde{\mu}_k$, one sees that (3.45) is equivalent to

$$\lim_{k\to\infty} k^{1-d}\Upsilon^\ell \left| \sum_{n=1}^{\infty} [(\theta_n(k))^\ell - (\sin\theta_n(k))^\ell] \right| = 0. \tag{3.46}$$

By (1.22), for $\delta \in [0,1)$ we have $0 < |\theta_n(k)| < \pi/4$ for all sufficiently large k and all n. From the elementary estimates $|\theta_n| \leq 2|\sin\theta_n|$ and $|\theta_n - \sin\theta_n| \leq C|\sin\theta_n|^3$ which hold for $|\theta_n| < \pi/4$, it follows that for all sufficiently large k

$$k^{1-d}\Upsilon^\ell \sum_{n=1}^{\infty} |(\theta_n)^\ell - (\sin\theta_n)^\ell| \leq k^{1-d}\Upsilon^\ell \sum_{n=1}^{\infty} \left(|\theta_n - \sin\theta_n| \sum_{j=0}^{\ell-1} |\theta_n|^j |\sin\theta_n|^{\ell-1-j} \right) \leq$$

$$\leq k^{1-d}\Upsilon^\ell C(\ell) \sum_{n=1}^{\infty} |\sin\theta_n|^{\ell+2} = k^{1-d}\Upsilon^\ell C(\ell) \|\mathrm{Im}\, S(k)\|_{\ell+2}^{\ell+2} = k^{-d-(1-2\delta)} C(\ell) \|\Upsilon \mathrm{Im}\, S(k)\|_{\ell+2}^{\ell+2}.$$

Now (3.46) follows by combining the estimate (3.16) for $\|\Upsilon \mathrm{Im}\, S(k)\|_{\ell+2}$ with the result just obtained and noting that $\delta \in [0,1)$. □

3.4. **Proof of Theorem 1.8.** By the estimate (1.22), for $\delta \in [0,1)$ the supports of $\tilde{\mu}_k$ are bounded uniformly in $k \geq 1$. On the other hand, by the boundedness of V, the support of μ^e is also bounded. Thus, we may choose $T > 0$ such that

$$\mathrm{supp}\, \mu^e \subset [-T, T] \quad \text{and} \quad \mathrm{supp}\, \tilde{\mu}_k \subset [-T,T] \quad \forall k \geq 1, \quad \delta \in [0,1).$$

Next, fix $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$, and let ℓ_0 be an even natural number satisfying $\ell_0 > \frac{d-1}{\rho-1}$. Since $\varphi(t)$ vanishes near $t = 0$, the function $\varphi(t)/t^{\ell_0}$ is smooth. By the Weierstrass approximation theorem, for any $\varepsilon > 0$ there exists a polynomial $\varphi_0(t)$ such that

$$|\varphi(t)t^{-\ell_0} - \varphi_0(t)| \leq \varepsilon, \quad \forall t \in [-T, T].$$

Denoting $\varphi_\pm(t) = (\varphi_0(t) \pm \varepsilon)t^{\ell_0}$, we obtain

$$\varphi_-(t) \leq \varphi(t) \leq \varphi_+(t), \quad \forall t \in [-T,T], \tag{3.47}$$

$$\varphi_+(t) - \varphi_-(t) = 2\varepsilon t^{\ell_0}. \tag{3.48}$$

By (3.47), we get

$$\int_{-\infty}^{\infty} \varphi_-(t)\mathrm{d}\tilde{\mu}_k(t) \leq \int_{-\infty}^{\infty} \varphi(t)\mathrm{d}\tilde{\mu}_k(t) \leq \int_{-\infty}^{\infty} \varphi_+(t)\mathrm{d}\tilde{\mu}_k(t). \tag{3.49}$$

By construction, $\varphi_\pm(t)$ are polynomials which involve powers t^m with $m \geq \ell_0$. Thus, we can apply Lemma 3.9 to (3.49), which yields

$$\limsup_{k\to\infty} k^{1-d} \int_{-\infty}^{\infty} \varphi(t)\mathrm{d}\tilde{\mu}_k(t) \leq \int_{-\infty}^{\infty} \varphi_+(t)\mathrm{d}\mu^e(t),$$
$$\liminf_{k\to\infty} k^{1-d} \int_{-\infty}^{\infty} \varphi(t)\mathrm{d}\tilde{\mu}_k(t) \geq \int_{-\infty}^{\infty} \varphi_-(t)\mathrm{d}\mu^e(t).$$

On the other hand, by (3.47), (3.48),

$$\int_{-\infty}^{\infty} \varphi_-(t)\mathrm{d}\mu^e(t) \leq \int_{-\infty}^{\infty} \varphi(t)\mathrm{d}\mu^e(t) \leq \int_{-\infty}^{\infty} \varphi_+(t)\mathrm{d}\mu^e(t)$$

and

$$\int_{-\infty}^{\infty} \varphi_+(t)\mathrm{d}\mu^e(t) - \int_{-\infty}^{\infty} \varphi_-(t)\mathrm{d}\mu^e(t) = 2\varepsilon \int_{-\infty}^{\infty} t^{\ell_0}\mathrm{d}\mu^e(t).$$

By (1.18), the integral in the right hand side of the last estimate is finite; denote this integral by C. Combining the above estimates, we obtain

$$\limsup_{k\to\infty} k^{1-d} \int_{-\infty}^{\infty} \varphi(t)\mathrm{d}\tilde{\mu}_k(t) \leq \int_{-\infty}^{\infty} \varphi(t)\mathrm{d}\mu^e(t) + 2\varepsilon C,$$
$$\liminf_{k\to\infty} k^{1-d} \int_{-\infty}^{\infty} \varphi(t)\mathrm{d}\tilde{\mu}_k(t) \geq \int_{-\infty}^{\infty} \varphi(t)\mathrm{d}\mu^e(t) - 2\varepsilon C.$$

Since $\varepsilon > 0$ can be taken arbitrary small, we obtain the required statement. $\square$

4. Proof of Theorem 1.9

For the magnetic case, that is $H_0 = -\Delta$ and $H = (i\nabla + A)^2 + V(x)$ with $A \neq 0$, the Born approximation is no longer valid, and instead we turn to work by D. Yafaev in [31], which concerns an approximation at high energy to the corresponding scattering matrix $S(k)$. The approximation is derived from approximate solutions to the stationary Schrödinger equation $Hu = k^2 u$. Using Yafaev's approximation, we express the scattering matrix as a semiclassical ΨDO on the sphere with semiclassical parameter $h = k^{-1}$ and symbol $e^{iM(\omega,\xi)}$. The result of Theorem 1.9 follows by using Proposition 2.4 to prove that for any integers $\ell_1 \geq 0$, $\ell_2 \geq 0$ such that $\ell_1 + \ell_2 > \frac{d-1}{m}$, $m = \min\{1, \rho - 1\}$,

$$\lim_{k\to\infty} k^{1-d}\,\mathrm{Tr}[(S(k) - I)^{\ell_1}(S(k)^* - I)^{\ell_2}] = \int_{\mathbb{T}} (z-1)^{\ell_1}(\overline{z}-1)^{\ell_2}\mathrm{d}\mu^m(z)$$

where the measure μ^m is given in (1.29). The result then follows by an application of the Weierstrass approximation theorem.

4.1. **Approximate solutions to the Schrödinger equation.**

Here we recall the construction of approximate solutions to the Schrödinger equation $Hu = k^2 u$ from [31] where V and A are both infinitely smooth and satisfy the estimates (1.1). The solutions u are sought as functions

$$u = u(x,p), \quad x \in \mathbb{R}^d, \quad p \in \mathbb{R}^d, \quad |p| = k.$$

We denote $\hat{p} = p|p|^{-1} \in \mathbb{S}^{d-1}$. We set

$$u(x,p) = e^{i\Theta(x,p)}\upsilon(x,p),$$

where the functions υ and Θ are to be determined. We note the Schrödinger operator H is equal to

$$H = -\Delta + 2i\langle A(x), \nabla\rangle + i\,\mathrm{div}\,A(x) + |A(x)|^2 + V(x).$$

We now plug the function u into the Schrödinger equation

$$Hu = k^2 u.$$

By direct calculation,

$$\begin{aligned} 2i\langle A(x), \nabla\rangle u &= -2\langle A, \nabla\Theta\rangle u + 2ie^{i\Theta}\langle A, \nabla\upsilon\rangle, \\ -\Delta u &= -(i\Delta\Theta)u + |\nabla\Theta|^2 u - ie^{i\Theta}\nabla\upsilon - ie^{i\Theta}\langle\nabla\Theta, \nabla\upsilon\rangle - e^{i\Theta}\Delta\upsilon, \end{aligned}$$

and hence we obtain

$$\begin{aligned} &[|\nabla\Theta|^2 - 2\langle A, \nabla\Theta\rangle + |A(x)|^2 + V(x)]u + \\ &\quad + e^{i\Theta}[-2i\langle\nabla\Theta, \nabla\upsilon\rangle + 2i\langle A, \nabla\upsilon\rangle - \Delta\upsilon + (-i\Delta\Theta + i\,\mathrm{div}\,A)\upsilon] = k^2 u. \end{aligned}$$

As a consequence, we obtain the eikonal equation

$$|\nabla\Theta|^2 - 2\langle A, \nabla\Theta\rangle + (V(x) + |A(x)|^2) = k^2 \tag{4.1}$$

for the phase function Θ and the transport equation

$$-2i\langle\nabla\Theta,\nabla\upsilon\rangle+2i\langle A,\nabla\upsilon\rangle-\Delta\upsilon+(-i\Delta\Theta+i\operatorname{div}A)\upsilon=0 \tag{4.2}$$

for the amplitude function υ. We seek the function Θ in the form

$$\Theta_\pm(x,p)=\langle x,p\rangle+\Phi_\pm(x,p) \tag{4.3}$$

where we solve the eikonal equation by iterations. We set for $N_0\in\mathbb{N}$

$$\Phi_\pm(x,p)=\Phi_\pm^{(N_0)}(x,p)=\sum_{n=0}^{N_0}(2k)^{-n}\phi_n^\pm(x,\hat p) \tag{4.4}$$

and plug (4.3),(4.4) into the eikonal equation (4.2). By comparing coefficients at the same powers of $(2k)^{-n}$, $n=-1,0,\dots,N_0-1$, the following solution is obtained:

$$\langle\hat p,\nabla\phi_0^\pm\rangle=\langle\hat p,A\rangle,\quad n=-1, \tag{4.5}$$

$$\langle\hat p,\nabla\phi_1^\pm\rangle+|\nabla\phi_0^\pm|^2-2\langle A,\nabla\phi_0^\pm\rangle+|A(x)|^2+V(x)=0,\quad n=0, \tag{4.6}$$

$$\langle\hat p,\nabla\phi_{n+1}^\pm\rangle+\sum_{m=0}^{n}\langle\nabla\phi_m^\pm,\nabla\phi_{m-n}^\pm\rangle-2\langle A,\nabla\phi_n^\pm\rangle=0,\quad n\geq 1. \tag{4.7}$$

The error associated with the approximate solution $\Phi_\pm^{(N_0)}(x,p)$ is given by

$$q_0(x,p)=\sum_{n+m\geq N_0}(2k)^{-n-m}\langle\nabla\phi_n^\pm,\nabla\phi_m^\pm\rangle-2(2k)^{-N_0}\langle A,\nabla\phi_{N_0}^\pm\rangle.$$

The equations (4.5) - (4.7) all have the form

$$\langle\hat p,\nabla\phi_n^\pm(x,\hat p)\rangle+f_n(x,\hat p)=0 \tag{4.8}$$

which can be explicitly solved as

$$\phi_n^\pm(x,\hat p)=\pm\int_0^\infty f_n(x\pm t\hat p,\hat p)\mathrm{d}t.$$

We now solve the transport equation (4.2) by a similar method. For a given $N\in\mathbb{N}$, the approximate solution to the transport equation is constructed as the asymptotic series

$$\upsilon_\pm(x,p)=\upsilon_\pm^{(N)}(x,p)=\sum_{n=0}^{N}(2ik)^{-n}\tilde\upsilon_n^\pm(x,\hat p), \tag{4.9}$$

We plug this function into the transport equation and compare coefficients at the same powers of $(2ik)^{-n}$ to obtain the recurrence relations

$$\langle\hat p,\nabla\tilde\upsilon_{n+1}^\pm\rangle=2i\langle A-\nabla\Phi_\pm,\nabla\tilde\upsilon_n^\pm\rangle-\Delta\tilde\upsilon_n^\pm+(-i\Delta\Phi_\pm+i\operatorname{div}A+q_0)\tilde\upsilon_n^\pm,\quad n=0,1,\dots,N.$$

Again all the above equations have the form

$$\langle\hat p,\nabla\tilde\upsilon_{n+1}^\pm(x,p)\rangle+f_{n+1}(x,p)=0$$

which may be solved explicitly as

$$\tilde{\upsilon}_{n+1}^{(\pm)}(x,\hat{p}) = \mp \int_0^\infty f_{n+1}^{(\pm)}(x + t\hat{p}, \hat{p})\mathrm{d}t,$$

where

$$f_n^{(\pm)} = 2i\langle A - \nabla\Phi_\pm, \nabla\tilde{\upsilon}_n^{(\pm)}\rangle - \Delta\tilde{\upsilon}_n^{(\pm)} + (|\nabla\Phi_\pm|^2 - 2\langle A, \nabla\Phi_\pm\rangle + V_1 - i\Delta\Phi_\pm)\tilde{\upsilon}_n^{(\pm)},$$

and

$$V_1 = V + |A|^2 + i\,\mathrm{div}\,A.$$

Let us now recall the definition of the function u. We may write $u = e^{i\Theta(x,p)}\upsilon(x,p)$ where

$$\Theta_\pm(x,p) = \langle x, p\rangle + \phi_\pm(x,\hat{p}). \tag{4.10}$$

The function $\phi_\pm(x,\hat{p})$ is just the first term of $\Phi_\pm$ given in (4.4), that is

$$\phi_\pm(x,\hat{p}) = \phi_0^\pm(x,\hat{p}) = \mp \int_0^\infty \langle A(x \pm t\hat{p}), \hat{p}\rangle \mathrm{d}t. \tag{4.11}$$

The rest of the terms in (4.4) tend to zero for large k. By expanding the exponential function containing such terms in a power series, it may be seen that they can be absorbed into the definition of $\upsilon_\pm(x,p)$ given by (4.9). Note in particular, $\upsilon_0^\pm(x,\hat{p}) \equiv 1$. This is the form of u we shall consider. When considering the functions $\phi_\pm$ and $\upsilon_n^\pm$, we will always exclude a conical neighbourhood of the direction $\hat{x} = -\hat{p}$ (for the sign "$+$") or $\hat{x} = \hat{p}$ (for the sign "$-$"). That is, we shall consider them only in the region $\Gamma_\pm(\varepsilon, R) \subset \mathbb{R}^d \times \mathbb{R}^d$ distinguished by the following condition: $(x,p) \in \Gamma_\pm(\varepsilon, R)$ if either $|x| \le R$ or $\pm\langle \hat{x}, \hat{p}\rangle \ge -1 + \varepsilon$ for some $\varepsilon > 0$ and $R > 0$. Outside these neighbourhoods, the functions $\phi_\pm$ and $\upsilon_n^\pm$, $n \ge 1$, decay at infinity in the x-variable and provide good approximations to the eikonal and transport equations. More precisely, the following statement is proven in [31]:

Proposition 4.1. [31] *Let both the electric potential V and the magnetic vector-potential A be infinitely differentiable and satisfy the estimates* (1.1). *Let $(x,\omega) \in \mathbb{R}^d \times \mathbb{S}^{d-1}$ be such that $(x,\omega) \in \Gamma_\pm(\varepsilon, R)$ for some $\varepsilon > 0$ and $R > 0$. Then the functions $\phi_\pm$ and $\upsilon_n^\pm$, $n \ge 1$, satisfy the estimates*

$$|\partial_x^\alpha \partial_\omega^\beta \phi_\pm(x,\omega)| \le C_{\alpha\beta}(\varepsilon, R)\langle x\rangle^{1-\rho-|\alpha|}, \tag{4.12}$$

$$|\partial_x^\alpha \partial_\omega^\beta \upsilon_n^\pm(x,\omega)| \le C_{\alpha\beta}(\varepsilon, R)\langle x\rangle^{-n-|\alpha|}, \tag{4.13}$$

for all multi-indices α, β.

We will write

$$u_\pm^{(N)}(x,p) = u_\pm(x,p) = e^{i\Theta_\pm(x,p)}\upsilon_\pm^{(N)}(x,p). \tag{4.14}$$

4.2. **Approximation to the scattering matrix.** Here we recall the approximation to the scattering matrix obtained in [31]. It is known (see [1]) that off the diagonal $\omega = \omega'$, the integral kernel $s(\omega, \omega'; k)$ of the scattering matrix $S(k)$ is a C^∞-smooth function of $\omega, \omega' \in \mathbb{S}^{d-1}$ and it tends to zero faster than any power of k^{-1} as $k \to \infty$. Thus, it suffices to describe the structure of $s(\omega, \omega'; k)$ in a neighbourhood of the diagonal $\omega = \omega'$. Let $\omega_0 \in \mathbb{S}^{d-1}$ be an arbitrary point and for $\delta \in (0, 1)$, let $\Omega = \Omega(\omega_0, \delta) \subset \mathbb{S}^{d-1}$ be the conical neighbourhood of ω_0 given by

$$\Omega(\omega_0, \delta) = \{\omega \in \mathbb{S}^{d-1} : \langle \omega, \omega_0 \rangle > \delta\}. \tag{4.15}$$

Let $u_\pm$ be as in (4.14). We set

$$x = \omega_0 z + y, \quad z \in \mathbb{R}, \quad y \in \Lambda_{\omega_0},$$

and

$$(\nabla_{\omega_0} u)(x, p) = \frac{\partial}{\partial z} u(\omega_0 z + y, p).$$

For $\omega, \omega' \in \Omega$, define

(4.16)

$$s_0^{(N)}(\omega, \omega'; k) = -i\pi k^{d-2} (2\pi)^{-d} \times$$
$$\times \left(\int_{\Lambda_{\omega_0}} \left[\overline{u_+^{(N)}(x, k\omega)} (\nabla_{\omega_0} u_-^{(N)})(x, k\omega') - \overline{(\nabla_{\omega_0} u_+^{(N)})(x, k\omega)} u_-^{(N)}(x, k\omega') \right] \mathrm{d}x - \right.$$
$$\left. - 2i \int_{\Lambda_{\omega_0}} \langle A(x), \omega_0 \rangle \overline{u_+^{(N)}(x, k\omega)} u_-^{(N)}(x, k\omega') \mathrm{d}x \right).$$

The integrals in (4.16) do not converge absolutely and should be understood as oscillatory integrals. In other words, (4.16) should be understood as a distribution on $\Omega \times \Omega$.

Proposition 4.2. [31] *For any $q \in \mathbb{N}$ there exists $N = N(q) \in \mathbb{N}$ such that for any $\omega_0 \in \mathbb{S}^{d-1}$, the kernel*

$$\widetilde{s}^{(N)}(\omega, \omega'; k) = s(\omega, \omega'; k) - s_0^{(N)}(\omega, \omega'; k)$$

belongs to the class $C^q(\Omega \times \Omega)$ and its C^q-norm is $O(k^{-q})$ as $k \to \infty$.

4.3. **The scattering matrix as a ΨDO on the sphere.** Below we represent the scattering matrix $S(k)$ as a semiclassical ΨDO on the sphere with semiclassical parameter $h = k^{-1}$. The statements almost identical to Proposition 4.3 can be found in [31, Propositions 6.1 and 6.4] and [30, Section 8.4]. Note that Proposition 4.3 is similar to Proposition 3.6 where the imaginary part of the Born approximation for a smooth, compactly supported potential V was also expressed as a semiclassical ΨDO on the sphere with semiclassical parameter $h = k^{-1}$. There, the explicit form of the Born approximation was used to obtain the representation, whereas here we use the kernel function (4.16) derived by Yafaev.

Proposition 4.3. *Suppose that the electric potential V and the magnetic vector-potential A are both infinitely differentiable and satisfy the estimates* (1.1). *Let $m = \min\{1, \rho - 1\}$. Then for any $q \in \mathbb{N}$, the scattering matrix can be written as*

$$S(k) = I + \mathrm{Op}_{k^{-1}}[\sigma] + R_q(k), \tag{4.17}$$

where:

(i) *the symbol σ can be represented as*

$$\sigma = \sigma_0 + k^{-1}\sigma_1, \tag{4.18}$$

where $\sigma_0, \sigma_1 \in S^{-m}(T^*\mathbb{S}^{d-1})$ and

$$\sigma_0(\omega, \xi) = \exp(iM(\omega, \xi)) - 1; \tag{4.19}$$

(ii) *the operator $R_q(k)$ has an integral kernel in the class $C^q(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$ and its C^q-norm is $O(k^{-q})$ as $k \to \infty$.*

Proof. 1) Let $\phi, \psi \in C^\infty(\mathbb{S}^{d-1})$ be functions with disjoint supports. Then $\phi S(k)\psi$ has a C^∞-smooth integral kernel which decays faster than any power of k^{-1} as $k \to \infty$. The same comment applies to $\phi\,\mathrm{Op}_{k^{-1}}[a]\psi$ with $a \in S^{-m}(T^*\mathbb{S}^{d-1})$. This shows that using a sufficiently fine partition of unity on the sphere, one easily reduces the problem to approximating the integral kernel of $S(k)$ locally in any conical neighbourhood Ω of the form (4.15). Thus, we can use Proposition 4.2.

2) Let us rearrange the integrand in (4.16). Denote

$$w_\pm^{(N)}(x, p) = e^{i\phi_\pm(x,\hat{p})} v_\pm^{(N)}(x, p), \tag{4.20}$$

$$\widetilde{w}_\pm^{(N)}(x, p) = k e^{i\phi_\pm(x,\hat{p})} (v_\pm^{(N)}(x, p) - 1), \tag{4.21}$$

so that from (4.14),

$$u_\pm^{(N)}(x, k\omega) = e^{ik\langle x,\omega\rangle} w_\pm^{(N)}(x, k\omega) = e^{ik\langle x,\omega\rangle}(e^{i\phi_\pm(x,\omega)} + k^{-1}\widetilde{w}_\pm^{(N)}(x, k\omega)),$$

$$(\nabla_{\omega_0} u_\pm^{(N)})(x, k\omega) = e^{ik\langle x,\omega\rangle}[ik\langle\omega_0, \omega\rangle e^{i\phi_\pm(x,\omega)} + i\langle\omega_0, \omega\rangle\widetilde{w}_\pm^{(N)}(x, k\omega) + (\nabla_{\omega_0} w_\pm^{(N)})(x, k\omega)].$$

The variables ω and ω' belong to the set $\Omega(\omega_0, \delta)$ for some fixed choice of δ. Now some elementary algebra shows that formula (4.16) can be rewritten as

$$s_0^{(N)}(\omega, \omega'; k) = k^{d-1}(2\pi)^{1-d} \int_{\Lambda_{\omega_0}} e^{-ik\langle\omega-\omega',x\rangle} a^{(N)}(\omega, \omega', x)\mathrm{d}x, \tag{4.22}$$

where

$$a^{(N)}(\omega, \omega', x) = 2^{-1}\langle\omega_0, \omega + \omega'\rangle \exp[i\phi_-(x, \omega') - i\phi_+(x, \omega)] + k^{-1}a_1^{(N)}(\omega, \omega', x), \tag{4.23}$$

$$
\begin{aligned}
2a_1^{(N)}(\omega,\omega',x) = {}& \langle\omega,\omega_0\rangle(e^{i\phi_-(x,\omega')}\overline{\widetilde{w}_+^{(N)}(x,k\omega)} + \overline{w_+^{(N)}(x,k\omega)}\widetilde{w}_-^{(N)}(x,k\omega')) \\
&+\langle\omega',\omega_0\rangle(e^{-i\phi_+(x,\omega)}\widetilde{w}_-^{(N)}(x,k\omega') + \overline{\widetilde{w}_+^{(N)}(x,k\omega)}w_-^{(N)}(x,k\omega')) \\
&+i\overline{(\nabla_{\omega_0}w_+^{(N)})(x,k\omega)}w_-^{(N)}(x,k\omega') - i\overline{w_+^{(N)}(x,k\omega)}(\nabla_{\omega_0}w_-^{(N)})(x,k\omega') \\
&-2\langle A(x),\omega_0\rangle\overline{w_+^{(N)}(x,k\omega)}w_-^{(N)}(x,k\omega').
\end{aligned}
\tag{4.24}
$$

Note that the choice $a^{(N)}(\omega,\omega',x) = \frac{1}{2}\langle\omega+\omega',\omega_0\rangle$ in (4.22) yields a δ-function on the sphere. Thus, we can write

(4.25)
$$
s_0^{(N)}(\omega,\omega';k) - \delta(\omega-\omega') = k^{d-1}(2\pi)^{1-d}\int_{\Lambda_{\omega_0}} e^{-ik\langle\omega-\omega',x\rangle}(a_0 + k^{-1}a_1^{(N)})(\omega,\omega',x)\mathrm{d}x,
$$

where

$$
a_0(\omega,\omega',x) = 2^{-1}\langle\omega_0,\omega+\omega'\rangle[\exp(i\phi_-(x,\omega') - i\phi_+(x,\omega)) - 1]. \tag{4.26}
$$

3) Let us change variables in the integral (4.25). In step 2) of the proof of Proposition 3.6, we wrote $x\in\mathbb{R}^d$ in the form (3.30) so that the variable ξ was orthogonal to the difference $\omega-\omega'$. This motivates the following change of variables. Instead of integrating over $x\in\Lambda_{\omega_0}$, we shall integrate over $\xi\in\Lambda_\omega$, where

$$
x = \xi - \frac{\langle\xi,\omega_0\rangle}{\langle\omega+\omega',\omega_0\rangle}(\omega+\omega'). \tag{4.27}
$$

Recall that $\omega,\omega'\in\Omega(\omega_0,\delta)$ and hence the denominator of (4.27) does not vanish. An easy calculation shows that

$$
\langle x,\omega-\omega'\rangle = \langle\xi,\omega-\omega'\rangle.
$$

Thus, we obtain

$$
s_0^{(N)}(\omega,\omega';k) - \delta(\omega-\omega') = k^{d-1}(2\pi)^{1-d}\int_{\Lambda_\omega} e^{-ik\langle\omega-\omega',\xi\rangle}b(\omega,\omega',\xi)\mathrm{d}\xi, \tag{4.28}
$$

where $b = b_0 + k^{-1}b_1$ with

$$
b_j(\omega,\omega',\xi) = J(\omega,\omega')a_j(\omega,\omega',x(\xi)) \tag{4.29}
$$

and $J(\omega,\omega')$ denotes the Jacobian of the linear map (4.27) considered as a map from Λ_ω to Λ_{ω_0}. It is easy to see that $J(\omega,\omega')$ is a smooth function of $\omega,\omega'\in\Omega$. We shall need the value of $J(\omega,\omega')$ on the diagonal $\omega=\omega'$, in which case the change of variables (4.27) is given by

$$
x = t\langle\omega,\omega_0\rangle^{-1}\omega + \xi, \quad t = -\langle\xi,\omega_0\rangle.
$$

Let us define a basis for Λ_{ω_0} and Λ_ω as follows. Let $(x_1,\dots,x_{d-2})\subset\mathbb{R}^{d-2}$. Let us define a vector x' as a projection of ω onto Λ_{ω_0} by

$$
x' = \omega - \langle\omega,\omega_0\rangle^{-1}\omega_0
$$

where we note that $\langle \omega, \omega_0 \rangle > 0$. We choose the basis of Λ_{ω_0} to be $(x_1, \ldots, x_{d-2}, x')$. Next, we define a vector x'' as a projection of ω_0 onto Λ_w by

$$x'' = \omega_0 - \langle \omega, \omega_0 \rangle \omega$$

and choose the basis of Λ_ω to be $(x_1, \ldots, x_{d-2}, x'')$. Note that $x' = -\langle \omega, \omega_0 \rangle^{-1} x''$. Then the Jacobian $J(\omega, \omega)$ is given by a lower triangular matrix with diagonal $(1, \ldots, 1, -\langle \omega, \omega_0 \rangle^{-1})$. Since the Jacobian matrix is lower triangular, the Jacobian determinant is given by the absolute value of the product of the diagonal entries, that is

$$J(\omega, \omega) = \langle \omega, \omega_0 \rangle^{-1}. \tag{4.30}$$

4) The right hand side of equation (4.28) is a semiclassical ΨDO with the amplitude b, see (2.8). In order to complete the proof, by Proposition 2.5 if suffices to check the estimates

$$|\partial_\xi^\alpha \partial_\omega^\beta \partial_{\omega'}^\gamma b_j(\omega, \omega', \xi)| \leq C_{\alpha\beta\gamma} \langle \xi \rangle^{-m-|\alpha|}, \tag{4.31}$$

for $j = 0, 1$ and all multi-indices α, β, γ uniformly over $k \geq 1$, and to check the identity

$$b_0(\omega, \omega, \xi) = \exp(iM(\omega, \xi)) - 1. \tag{4.32}$$

Let us first check (4.32). Recalling the definition (1.27) of M and the definition (4.11) of $\phi_\pm$, we get

$$M(\omega, \xi) = \phi_-(\xi, \omega) - \phi_+(\xi, \omega).$$

From this and (4.26), (4.29) and (4.30),

$$b_0(\omega, \omega, \xi) = J(\omega, \omega) a_0(\omega, \omega, x(\xi)) = \exp(iM(\omega, x(\xi))) - 1,$$

where $x(\xi)$ is the linear map (4.27). Next, by the definition of the map $x(\xi)$, for $\omega = \omega'$ it takes the form $x(\xi) = \xi + c\omega$, and by the definition of the function M we have

$$M(\omega, \xi + c\omega) = M(\omega, \xi).$$

Thus, we obtain (4.32).

5) It remains to check that the estimates (4.31) are satisfied. This is essentially a consequence of Proposition 4.1; let us check this. Recall that $m = \min\{1, \rho - 1\}$. In what follows we make reference to Appendix B, and we denote by α and β multi-indices. Firstly, it is clear from (4.12) that $\phi_\pm(x, \omega) \in S^{-m}(T^*\mathbb{S}^{d-1})$, so that

$$|\partial_x^\alpha \partial_\omega^\beta \phi_\pm(x, \omega)| \leq C_{\alpha\beta} \langle x \rangle^{-m-|\alpha|}. \tag{4.33}$$

Next let us consider $\partial_x^\alpha \partial_\omega^\beta \widetilde{w}_\pm(x, k\omega)$. It follows from the fact that $\phi_\pm(x,\omega) \in S^{-m}(T^*\mathbb{S}^{d-1})$ and Lemma B.2 that the function $e^{i\phi_\pm(x,k\omega)} \in S^0(T^*\mathbb{S}^{d-1})$. Further,

$$\upsilon_\pm^{(N)}(x, k\omega) - 1 = \sum_{n=1}^{N} (2ik)^{-n} \upsilon_n^\pm(x, k\omega)$$

where the functions $\upsilon_n^\pm$ satisfy the estimates (4.13). As a result, the function $k(\upsilon_\pm^{(N)}(x, k\omega) - 1) \in S^{-1}(T^*\mathbb{S}^{d-1})$. Therefore by Lemma B.1,

$$|\partial_x^\alpha \partial_\omega^\beta \widetilde{w}_\pm(x, k\omega)| \le C_{\alpha\beta} \langle x \rangle^{-m-|\alpha|}. \tag{4.34}$$

We next consider $\partial_x^\alpha \partial_\omega^\beta (\nabla_{\omega_0} w_\pm)(x, k\omega)$. Again it follows from Lemma B.2 that

$$|\partial_x^\alpha \partial_\omega^\beta (\nabla_{\omega_0} w_\pm)(x, k\omega)| \le C_{\alpha\beta} \langle x \rangle^{-m-|\alpha|}. \tag{4.35}$$

Finally, we consider the function $w_\pm(x, k\omega)$, which we write as

$$w_\pm(x, k\omega) = e^{i\phi_\pm(x,k\omega)} + k^{-1} \widetilde{w}_\pm(x, k\omega).$$

Since $e^{i\phi_\pm(x,k\omega)} \in S^0(T^*\mathbb{S}^{d-1})$ by Lemma B.2, it follows from (4.34) that

$$|\partial_x^\alpha \partial_\omega^\beta w_\pm(x, k\omega)| \le C_{\alpha\beta} \langle x \rangle^{-|\alpha|}. \tag{4.36}$$

All of the above estimates are uniform in $k \ge 1$. It follows from the estimates (4.33) - (4.36) together with Lemma B.1 that the function $a_1(\omega, \omega', x)$ defined in (4.24) satisfies

$$|\partial_x^\alpha \partial_\omega^\beta \partial_{\omega'}^\gamma a_1(\omega, \omega', x)| \le C_{\alpha\beta} \langle x \rangle^{-m-|\alpha|}, \tag{4.37}$$

uniformly in $k \ge 1$. Next, we recall that $\phi_\pm(x, \omega) \in S^{-m}(T^*\mathbb{S}^{d-1})$. Therefore it follows easily from Lemma B.3 that the function $a_0(\omega, \omega', x)$ defined by (4.26) satisfies

$$|\partial_x^\alpha \partial_\omega^\beta \partial_{\omega'}^\gamma a_0(\omega, \omega', x)| \le C_{\alpha\beta} \langle x \rangle^{-m-|\alpha|}, \tag{4.38}$$

uniformly in $k \ge 1$. Now from (4.37), (4.38), (4.29) and (4.27), by an elementary calculation we obtain (4.31). $\square$

4.4. **The case of a monomial φ.**

Lemma 4.4. *Suppose that the electric potential V and the magnetic vector-potential A are both infinitely differentiable and satisfy the estimates* (1.1)*. Then for any integers $\ell_1 \ge 0$, $\ell_2 \ge 0$ such that $\ell_1 + \ell_2 > (d-1)/m$, $m = \min\{1, \rho - 1\}$, one has*

$$\lim_{k\to\infty} k^{1-d} \operatorname{Tr}[(S(k) - I)^{\ell_1} (S(k)^* - I)^{\ell_2}] = \int_{\mathbb{T}} (z-1)^{\ell_1} (\overline{z} - 1)^{\ell_2} \mathrm{d}\mu^m(z) \tag{4.39}$$

where the measure μ^m is given in (1.29).

Remark. This lemma is similar in structure to that of Lemma 3.7. There, similar asymptotics to (4.39) were determined for $\operatorname{Im} S_B(k)$. However here we have to consider the products $(S(k) - I)^{\ell_1} (S(k)^* - I)^{\ell_2}$. This is due to the use of the Weierstrass approximation theorem and is explained further in Section 4.5.

Proof. By Lemma 4.3, we have

$$(S(k)-I)^{\ell_1}(S(k)^*-I)^{\ell_2}=(\mathrm{Op}_{k^{-1}}[\sigma]+R_q(k))^{\ell_1}\big((\mathrm{Op}_{k^{-1}}[\sigma])^*+R_q(k)^*\big)^{\ell_2}, \tag{4.40}$$

where σ, $R_q(k)$ are as described in Lemma 4.3. Expanding the brackets in (4.40), we obtain

$$(S(k)-I)^{\ell_1}(S(k)^*-I)^{\ell_2}=(\mathrm{Op}_{k^{-1}}[\sigma])^{\ell_1}\big((\mathrm{Op}_{k^{-1}}[\sigma])^*\big)^{\ell_2}+Q_q(k), \tag{4.41}$$

where $Q_q(k)$ is the sum of the products of operators, to be estimated below. For the first term in the r.h.s in (4.41), by Proposition 2.4, we have

$$\lim_{k\to\infty}\left(\frac{k}{2\pi}\right)^{-d+1}\mathrm{Tr}\big((\mathrm{Op}_{k^{-1}}[\sigma])^{\ell_1}\big((\mathrm{Op}_{k^{-1}}[\sigma])^*\big)^{\ell_2}\big)$$
$$=\int_{\mathbb{S}^{d-1}}\int_{\Lambda_\omega}\big(e^{iM(\omega,\xi)}-1\big)^{\ell_1}\big(e^{-iM(\omega,\xi)}-1\big)^{\ell_2}\mathrm{d}\xi\mathrm{d}\omega.$$

Recalling the definition of the measure μ^m, we see that the right hand side of the above is equivalent to the right hand side of (4.39). Let us check that by a suitable choice of q we can ensure that the error term $\mathrm{Tr}\, Q_q(k)$ remains bounded as $k\to\infty$; this will yield the desired asymptotics (4.39). By choosing q sufficiently large, if follows from Proposition A.4 that we can make sure the estimate ($\|\cdot\|_1$ is the trace norm)

$$\|R_q(k)\|_1=O(1),\quad k\to\infty, \tag{4.42}$$

holds true. Next, using the estimates

$$|\mathrm{Tr}(AB)|\le\|A\|\|B\|_1 \text{ and } \|C\|\le\|C\|_1$$

for trace class operators B and C and bounded operator A and recalling that $Q_q(k)$ arose as a remainder term in the expansion of the brackets in the l.h.s. of (4.40), we obtain

$$|\mathrm{Tr}(Q_q(k))|\le C(\ell_1,\ell_2)\|R_q(k)\|_1\,\|\mathrm{Op}_{k^{-1}}[\sigma]\|^{\ell_1+\ell_2-1}. \tag{4.43}$$

By (2.3), we have

$$\|\mathrm{Op}_{k^{-1}}[\sigma]\|=O(1),\quad k\to\infty.$$

Combining the last inequality with (4.42), we obtain that $\mathrm{Tr}(Q_q(k))$ is bounded as $k\to\infty$, as required. □

4.5. **Proof of Theorem 1.9.** It now remains to extend Lemma 4.4 to all permissible functions φ. We begin by stating a version of the Weierstrass approximation theorem concerning polynomials on the circle in the complex plane. We then state and prove a lemma which uses this result. The proof of Theorem 1.9 is a trivial consequence of this lemma.

The following result is well known (see e.g. [22] Section 4.24, and in particular Theorem 4.25)

Proposition 4.5 (Weierstrass approximation theorem)**.** *The set of polynomials on the unit circle is dense in $C(\mathbb{T})$.*

Lemma 4.6. *Let ℓ_0 be an even natural number and let ν, ν_k $(k \geq 1)$ be σ-finite measures on $\mathbb{T} \setminus \{1\}$ such that*

$$\int_{\mathbb{T}} |z-1|^{\ell_0} \mathrm{d}\nu(z) < \infty, \quad \int_{\mathbb{T}} |z-1|^{\ell_0} \mathrm{d}\nu_k(z) < \infty, \tag{4.44}$$

for all k. Suppose that for all integers $\ell_1 \geq 0$, $\ell_2 \geq 0$ such that $\ell_1 + \ell_2 \geq \ell_0$, the relation

$$\lim_{k\to\infty} \int_{\mathbb{T}} (z-1)^{\ell_1} (\overline{z}-1)^{\ell_2} d\nu_k(z) = \int_{\mathbb{T}} (z-1)^{\ell_1} (\overline{z}-1)^{\ell_2} d\nu(z) \tag{4.45}$$

holds true. Then for any $\varphi \in C(\mathbb{T})$ such that $\varphi(z)|z-1|^{-\ell_0}$ is continuous on $\mathbb{T}$, we have

$$\lim_{k\to\infty} \int_{\mathbb{T}} \varphi(z) \mathrm{d}\nu_k(z) = \int_{\mathbb{T}} \varphi(z) \mathrm{d}\nu(z). \tag{4.46}$$

Proof. Applying Proposition 4.5 to the continuous function $\varphi(z)|z-1|^{-\ell_0}$, for any $\varepsilon > 0$ we obtain a polynomial $\varphi_0(z)$ in z, $\overline{z}$ such that

$$|\varphi(z)|z-1|^{-\ell_0} - \varphi_0(z)| \leq \varepsilon, \quad \forall z \in \mathbb{T}.$$

Let us define $\varphi_\pm(z) = (\operatorname{Re} \varphi_0(z) \pm \varepsilon)|z-1|^{\ell_0}$, then it follows from the above that

$$\varphi_-(z) \leq \operatorname{Re} \varphi(z) \leq \varphi_+(z) \quad \forall z \in \mathbb{T}, \tag{4.47}$$

$$\varphi_+(z) - \varphi_-(z) = 2\varepsilon |z-1|^{\ell_0}. \tag{4.48}$$

By the construction of $\varphi_\pm$, it can be represented as a polynomial in $w = z-1$, $\overline{w} = \overline{z}-1$ involving only products $w^{\ell_1}\overline{w}^{\ell_2}$ with $\ell_1 + \ell_2 \geq \ell_0$. Thus, by (4.47), (4.48) we can write

$$\int_{\mathbb{T}} \varphi_-(z) \mathrm{d}\nu(z) \leq \int_{\mathbb{T}} \operatorname{Re} \varphi(z) \mathrm{d}\nu(z) \leq \int_{\mathbb{T}} \varphi_+(z) \mathrm{d}\nu(z), \tag{4.49}$$

$$\int_{\mathbb{T}} \varphi_+(z) \mathrm{d}\nu(z) - \int_{\mathbb{T}} \varphi_-(z) \mathrm{d}\nu(z) = 2\varepsilon \int_{\mathbb{T}} |z-1|^{\ell_0} d\nu(z), \tag{4.50}$$

where all integrals are absolutely convergent by (4.44). Denote by C the value of the integral in the r.h.s. of (4.50). Similarly to (4.49), we get

$$\int_{\mathbb{T}} \varphi_-(z) \mathrm{d}\nu_k(z) \leq \int_{\mathbb{T}} \operatorname{Re} \varphi(z) \mathrm{d}\nu_k(z) \leq \int_{\mathbb{T}} \varphi_+(z) \mathrm{d}\nu_k(z) \tag{4.51}$$

for all $k \geq 1$. Now we can use (4.45) to pass to the limit in (4.51). This yields

$$\limsup_{k\to\infty} \int_{\mathbb{T}} \operatorname{Re} \varphi(z) \mathrm{d}\nu_k(z) \leq \int_{\mathbb{T}} \varphi_+(z) \mathrm{d}\nu(z) \leq \int_{\mathbb{T}} \varphi(z) \mathrm{d}\nu(z) + 2\varepsilon C,$$

$$\liminf_{k\to\infty} \int_{\mathbb{T}} \operatorname{Re} \varphi(z) \mathrm{d}\nu_k(z) \geq \int_{\mathbb{T}} \varphi_-(z) \mathrm{d}\nu(z) \geq \int_{\mathbb{T}} \varphi(z) \mathrm{d}\nu(z) - 2\varepsilon C.$$

Since $\varepsilon > 0$ may be taken arbitrary small, we obtain

$$\lim_{k\to\infty} \int_{\mathbb{T}} \operatorname{Re}\varphi(z)\mathrm{d}\nu_k(z) = \int_{\mathbb{T}} \operatorname{Re}\varphi(z)\mathrm{d}\nu(z).$$

Since the same argument can be applied to the imaginary part of φ, we obtain the required statement. $\square$

Proof of Theorem 1.9. Recall that the measures μ_k and μ^m are defined in (1.5) and (1.29) respectively. Hence the result (4.39) can be expressed as

$$\lim_{k\to\infty} k^{1-d} \int_{\mathbb{T}} (z-1)^{\ell_1}(\overline{z}-1)^{\ell_2} d\mu_k(z) = \int_{\mathbb{T}} (z-1)^{\ell_1}(\overline{z}-1)^{\ell_2} d\mu^m(z).$$

Now it remains to apply Lemma 4.6 with $\nu_k = k^{1-d}\mu_k$ and $\nu = \mu^m$. $\square$

Remark 4.7. As mentioned in Section 4.4, the difference between Lemma 4.4 and Lemma 3.7 is due to the use of the Weierstrass approximation theorem in Lemma 4.6 and Section 3.4. Note that the set of polynomials $\{t^\ell\}_{\ell\geq 0}$ are complete in the space $C([-T,T])$ for any $T > 0$, but the polynomials $\{(z-1)^\ell\}_{\ell\geq 0}$ are not complete in the space $C(\mathbb{T})$. By using instead the polynomials $\{(z-1)^{\ell_1}(\overline{z}-1)^{\ell_2}\}_{\ell_1,\ell_2\geq 0}$ we do obtain a complete system in $C(\mathbb{T})$ and the Weierstrass approximation theorem is therefore valid in this case.

5. Proof of Theorem 1.5 using Yafaev's expansion

We here provide another proof of Theorem 1.5 using the methods discussed in the proof of Theorem 1.9. Note that the Schrödinger operators (and hence the associated scattering matrices) as defined in Theorems 1.5 and 1.9 are equivalent when the magnetic vector-potential A is identically zero. By using the methods of Theorem 1.9, we may work directly with the scattering matrix as opposed to the Born approximation. We denote by $S(k)$ the scattering matrix associated with the Schrödinger operators $H_0 = -\Delta$, $H = -\Delta + V$ where V denotes the operator of multiplication by a real valued potential function $V(x)$. Let us now provide a statement of the main result of this section.

Proposition 5.1. *Suppose that the magnetic vector-potential $A \equiv 0$ and the electric potential V is infinitely differentiable and satisfies the estimate*

$$|V(x)| \leq C_\alpha \langle x \rangle^{-\rho-|\alpha|}, \quad \rho > 1. \tag{5.1}$$

Let $m = \min\{1, \rho - 1\}$. Then for any $q \in \mathbb{N}$, the scattering matrix can be written as

$$k[S(k) - I] = i \operatorname{Op}_{k^{-1}}[\sigma] + R_q(k), \tag{5.2}$$

where:

(i) *The symbol σ can be represented as*

$$\sigma = \sigma_0 + k^{-1}\sigma_1, \tag{5.3}$$

where $\sigma_0, \sigma_1 \in S^{-m}(T^*\mathbb{S}^{d-1})$ and

$$\sigma_0(\omega, \xi) = X(\omega, \xi); \tag{5.4}$$

(ii) *the operator $R_q(k)$ has an integral kernel in the class $C^q(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$ and its C^q-norm is $O(k^{-q})$ as $k \to \infty$.*

Indeed, the statement of Proposition 5.1 is the representation of the scattering matrix as a semiclassical ΨDO on the sphere with semiclassical parameter k, similarly to Proposition 4.3 and Propositon 3.6.

Then Theorem 1.5 follows from this statement by repeating (with minor modifications) arguments found elsewhere in the text, specifically Lemmas 4.4, 3.8 and 3.9 together with the arguments of Section 3.4.

We begin by recalling from Section 4.1 the approximate solutions to the Schrödinger equation $Hu = k^2 u$ when the magnetic vector-potential $A \equiv 0$ and the corresponding approximation to the scattering matrix. Using this approximation, we express the scattering matrix as a semiclassical ΨDO on the sphere with semiclassical parameter $h = k^{-1}$ and symbol $iX(\omega, \xi)$, where $X(\omega, \xi)$ is defined in (1.14).

5.1. **Approximation to the scattering matrix.** Let us briefly recall from Section 4.1 the approximate solutions to the Schrödinger equation $Hu = k^2 u$ when the magnetic vector-potential $A \equiv 0$. For each $N \in \mathbb{N}$, the approximate solution is written as

$$u_{\pm}^{(N)}(x,p) = u_{\pm}(x,p) = e^{i\Theta(x,p)} \upsilon_{\pm}^{(N)}(x,p), \quad x \in \mathbb{R}^d, \quad p \in \mathbb{R}^d, \quad |p| = k, \tag{5.5}$$

where the function $\upsilon_{\pm}^{(N)}$ is expressed as an asymptotic series

$$\upsilon_{\pm}^{(N)}(x,p) = \sum_{n=0}^{N} (2ik)^{-n} \upsilon_n^{\pm}(x,\hat{p}) \tag{5.6}$$

and we choose $\Theta(x,p)$ to be just the first term of (4.10) i.e.

$$\Theta(x,p) = \langle x, p \rangle.$$

In particular, we note that Proposition 4.1 holds for the functions $\upsilon_n^{\pm}$ and

$$\upsilon_0^{\pm}(x,\hat{p}) \equiv 1$$

$$\upsilon_1^{\pm}(x,\hat{p}) = \mp \int_0^{\infty} V(x \pm t\hat{p}) \mathrm{d}t. \tag{5.7}$$

By recalling the notation of Section 4.2, we may write the kernel of the approximation to the scattering matrix as

$$\begin{aligned} s_0^{(N)}(\omega,\omega';k) &= -i\pi k^{d-2}(2\pi)^{-d} \times \\ &\times \int_{\Lambda_{\omega_0}} \left[\overline{u_+(x,k\omega)} (\partial_z u_-)(x,k\omega') - \overline{(\partial_z u_+)(x,k\omega)} u_-(x,k\omega') \right] \mathrm{d}x \end{aligned} \tag{5.8}$$

for $\omega, \omega' \in \Omega$. In particular, Proposition 4.2 holds for the kernel (5.8)

5.2. **The scattering matrix as a ΨDO on the sphere.** We now provide the proof of Proposition 5.1 stated earlier.

Proof. We shall make reference to the proof of Proposition 4.3 wherever possible and highlight only the significant differences. By repeating the same argument 1) from that proof, we reduce the problem to approximating the integral kernel of $S(k)$ locally in any conical neighbourhood Ω, and hence we can use Proposition 4.2.
1) We rewrite the integral in (5.8). We rescale by the parameter k, and by a simple calculation we see that (5.8) may be expressed as

$$\begin{aligned} k{s_0}^{(N)}(\omega,\omega';k) &= (2i)^{-1} k^{d-1} (2\pi)^{1-d} \int_{\Lambda_{\omega_0}} e^{-ik\langle \omega-\omega', x\rangle} [ik\langle \omega+\omega', \omega_0\rangle \overline{\upsilon_+^{(N)}(x,k\omega)} \upsilon_-^{(N)}(x,k\omega') \\ &+ \overline{\upsilon_+^{(N)}(x,k\omega)} (\partial_z \upsilon_-^{(N)})(x,k\omega') - \upsilon_-^{(N)}(x,k\omega') \overline{(\partial_z \upsilon_+^{(N)})(x,k\omega)}] \mathrm{d}x. \end{aligned}$$

Let us denote by g the function

$$g(x,k\omega,k\omega') = k^2 [\overline{\upsilon_+^{(N)}(x,k\omega)} \upsilon_-^{(N)}(x,k\omega') - 1 - (2ik)^{-1} (\upsilon_1^-(x,\omega') - \upsilon_1^+(x,\omega))] \tag{5.9}$$

where the functions $\upsilon_1^\pm$ are explicitly stated in (5.6). Then we may instead write

$$ks_0^{(N)}(\omega,\omega';k) = k^{d-1}(2\pi)^{1-d}\int_{\Lambda_{\omega_0}} e^{-ik\langle\omega-\omega',x\rangle}a(\omega,\omega',x)\mathrm{d}x \tag{5.10}$$

where

$$a(\omega,\omega',x) = 2^{-1}k\langle\omega+\omega',\omega_0\rangle[1+(2ik)^{-1}(\upsilon_1^-(x,\omega')-\upsilon_1^+(x,\omega))]+k^{-1}a_1(\omega,\omega'x), \tag{5.11}$$

$$\begin{aligned} 2a_1(\omega,\omega',x) = g(x,k\omega,k\omega') + i\upsilon_-^{(N)}(x,k\omega')\overline{(\partial_z\upsilon_+^{(N)})(x,k\omega)}- \\ -\overline{i\upsilon_+^{(N)}(x,k\omega)}(\partial_z\upsilon_-^{(N)})(x,k\omega'). \end{aligned} \tag{5.12}$$

Note that the choice of $a(\omega,\omega',x) = \frac{1}{2}\langle\omega+\omega',\omega_0\rangle$ in (5.10) yields a δ-function on the sphere. Thus, we can write

$$k[s_0^{(N)}(\omega,\omega';k) - \delta(\omega-\omega')] = k^{d-1}(2\pi)^{1-d}\int_{\Lambda_{\omega_0}} e^{-ik\langle\omega-\omega',x\rangle}(a_0+k^{-1}a_1)(\omega,\omega',x)\mathrm{d}x \tag{5.13}$$

where

$$a_0(\omega,\omega',x) = \frac{1}{2}\langle\omega+\omega',\omega_0\rangle(2i)^{-1}(\upsilon_1^-(x,\omega')-\upsilon_1^+(x,\omega)). \tag{5.14}$$

2) By changing variables in (5.10) in the same manner as 3) in the proof of Proposition 4.3, we obtain

$$k[s_0^{(N)}(\omega,\omega';k) - \delta(\omega-\omega')] = k^{d-1}(2\pi)^{1-d}\int_{\Lambda_\omega} e^{-ik\langle\omega-\omega',\xi\rangle}b(\omega,\omega',\xi)\mathrm{d}\xi, \tag{5.15}$$

where $b = b_0 + k^{-1}b_1$ with

$$b_j(\omega,\omega',\xi) = J(\omega,\omega')a_j(\omega,\omega',x(\xi)) \tag{5.16}$$

and $J(\omega,\omega')$ denotes the Jacobian of the linear map (4.27) considered as a map from Λ_ω to Λ_{ω_0}. It is easy to see that $J(\omega,\omega')$ is a smooth function of $\omega,\omega'\in\Omega$ and

$$J(\omega,\omega) = \langle\omega,\omega_0\rangle^{-1}. \tag{5.17}$$

3) The right hand side of equation (5.15) is a semiclassical ΨDO with the amplitude b, see (2.8). In order to complete the proof, by Proposition 2.5 if suffices to check the estimates

$$|\partial_x^\alpha\partial_\omega^\beta\partial_{\omega'}^\gamma b_j(\omega,\omega',\xi)| \le C_{\alpha\beta\gamma}\langle\xi\rangle^{-m-|\alpha|}, \tag{5.18}$$

for $j=0,1$ and all multi-indices α,β,γ uniformly over $k\ge 1$, and to check the identity

$$b_0(\omega,\omega,\xi) = iX(\omega,\xi). \tag{5.19}$$

Let us first check (5.19). Recalling the definition (1.14) of $X(\omega,\xi)$ and the definition (5.6) of $\upsilon_1^\pm$, we get

$$iX(\omega,\xi) = (2i)^{-1}[\upsilon_1^-(\xi,\omega)-\upsilon_1^+(\xi,\omega)].$$

From this and (5.14), (5.16) and (5.17),

$$b_0(\omega,\omega,\xi) = J(\omega,\omega)a_0(\omega,\omega,x(\xi)) = iX(\omega,x(\xi)),$$

where $x(\xi)$ is the linear map (4.27). Next, by the definition of the map $x(\xi)$, for $\omega=\omega'$ it takes the form $x(\xi)=\xi+c\omega$, and by the definition of the function X we have

$$X(\omega,\xi+c\omega) = X(\omega,\xi).$$

Thus, we obtain (5.19).

4) It remains to check that the estimates (5.18) are satisfied. This is essentially a consequence of Proposition 4.1; let us check this. Recalling that $m=\min\{1,\rho-1\}$ and using the estimates (4.13), we obtain

$$\begin{aligned}|\partial_x^\alpha\partial_\omega^\beta\partial_{\omega'}^\gamma g(x,\omega,\omega')| &\leq C_{\alpha\beta\gamma}\langle x\rangle^{-m-1-|\alpha|},\\ k|\partial_x^\alpha\partial_\omega^\beta\partial_{\omega'}^\gamma \upsilon_\pm^{(N)}(x,k\omega)(\partial_z\upsilon_\mp^{(N)})(x,k\omega')| &\leq C_{\alpha\beta\gamma}\langle x\rangle^{-m-|\alpha|}.\end{aligned}$$

where all the estimates are uniform in $k\geq 1$. It follows that a_0 and a_1, defined by (5.14), (5.12) respectively, satisfy

$$|\partial_x^\alpha\partial_\omega^\beta\partial_{\omega'}^\gamma a_j(\omega,\omega',x)| \leq C_{\alpha\beta\gamma}\langle x\rangle^{-m-|\alpha|} \tag{5.20}$$

uniformly in $k\geq 1$. Now from (5.20), (5.17) and (5.16), by an elementary calculation we obtain (5.18). $\square$

Appendix A. Schatten-class operators

We begin by noting that all of the following information may be found in [4] Supplement 1.

Let $\mathcal{H}$ be a Hilbert space and let $T : \mathcal{H} \to \mathcal{H}$ be a compact linear operator. By the Hilbert Schmidt theorem, there exists a countable ordered set

$$\lambda_1(T^*T) \geq \lambda_2(T^*T) \geq \cdots \geq 0$$

with $\lambda_n(T^*T) \to 0$ as $n \to \infty$. The values $\lambda_n(T^*T)$ represent the eigenvalues of the operator T^*T. Let us define the singular numbers

$$s_n(T) = \sqrt{\lambda_n(T^*T)}$$

of T. For any $\ell \in [1, \infty)$, we define the Schatten ℓ-norm of T to be

$$\|T\|_\ell = \left(\sum_{n \geq 1} [s_n(T)]^\ell \right)^{\frac{1}{\ell}} .$$

The class S_∞ denotes the set of all compact operators on $\mathcal{H}$ with the usual norm

$$\|T\|_\infty = \|T\| = \max_{n \in \mathbb{N}} s_n(T).$$

In particular, if $1 \leq p < q \leq +\infty$, we have the inclusion $S_p \subset S_q$.

We denote by $\mathcal{L}(\mathcal{H})$ the space of all bounded linear operators on $\mathcal{H}$. Let us now describe some standard properties of the Schatten norm:

$$\begin{aligned} \|BA\|_\ell &\leq \|B\|\|A\|_\ell, \quad A \in S_\ell(\mathcal{H}), \quad B \in \mathcal{L}(\mathcal{H}), \\ \|AB\|_\ell &\leq \|B\|\|A\|_\ell, \quad A \in S_\ell(\mathcal{H}), \quad B \in \mathcal{L}(\mathcal{H}), \\ \|A\|_\ell &= \|A^*\|_\ell, \quad A \in S_\ell(\mathcal{H}). \end{aligned}$$

Next, let $\ell \in [1, \infty]$, and let $\ell_1 \in [1, \infty]$, $\ell_2 \in [1, \infty]$ be chosen such that $\ell^{-1} = \ell_1^{-1} + \ell_2^{-1}$. Then we have the following Holder-type inequality:

$$\|AB\|_\ell \leq \|A\|_{\ell_1} \|B\|_{\ell_2}, \quad A \in S_{\ell_1}(\mathcal{H}), \quad B \in S_{\ell_2}(\mathcal{H}). \tag{A.1}$$

We next describe two important Schatten classes. The first is $S_2(\mathcal{H})$, which corresponds to the space of all Hilbert Schmidt operators on $\mathcal{H}$. We have the following important proposition which enables us to calculate directly the Hilbert Schmidt norm.

Proposition A.1. *Let $\mathcal{H}_1 = L^2(M_1, \mathrm{d}\sigma_1)$, $\mathcal{H}_2 = L^2(M_2, \mathrm{d}\sigma_1)$, where M_i is a measure space and $\mathrm{d}\sigma_i$ the corresponding measure for $i = 1, 2$. Then $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a Hilbert-Schmidt operator if and only if there exists a function*

$$\tau(m_2, m_1) \in L^2(M_2 \times M_1, \mathrm{d}\sigma_2 \times \mathrm{d}\sigma_1)$$

called the integral kernel of T such that T is expressed via $\tau(m_2, m_1)$ in the form

$$(Tf)(m_2) = \int_{M_1} \tau(m_2, m_1) f(m_1) \mathrm{d}\sigma_1.$$

The function $\tau(m_2, m_1)$ is uniquely defined (to within values on a set of $(\mathrm{d}\sigma_2 \times \mathrm{d}\sigma_1)$-measure zero) and

$$\|T\|_2^2 = \int\int |\tau(m_2, m_1)|^2 \mathrm{d}\sigma_2 \mathrm{d}\sigma_1. \tag{A.2}$$

The second class of note is $S_1(\mathcal{H})$, which corresponds to the space of all trace class operators on $\mathcal{H}$. A trace class operator A is a linear operator of the form

$$A = \sum_{1 \leq j \leq N} B_j C_j, \quad B_j, C_j \in S_2(\mathcal{H}),$$

where N depends on A.

Definition A.2. Let $A \in S_1(\mathcal{H})$ and let $\{e_\alpha\}$ be an orthonormal basis in $\mathcal{H}$. Then the trace of A, $\operatorname{Tr} A$, is defined as

$$\operatorname{Tr} A = \sum_\alpha \langle A e_\alpha, e_\alpha \rangle.$$

The trace of A is independent of the choice of orthonormal basis $\{e_\alpha\}$.

We now give a formula for the trace of a trace class operator A in $L^2(X)$ where X is a measure space with measure $\mathrm{d}x$. Note that since A is trace class, A is also a Hilbert Schmidt operator. Hence A may be expressed in terms of the kernel $A(x, y) \in L^2(X \times X)$ as

$$Au(x) = \int A(x, y) u(y) \mathrm{d}y.$$

Then the trace of A may be expressed as

$$\operatorname{Tr} A = \int A(x, x) \mathrm{d}x.$$

Next, we state a proposition which describes the relationship between the trace and the eigenvalues for a self-adjoint trace class operator T.

Proposition A.3. *Let T be a compact self-adjoint operator with $\{\lambda_n\}_{n \in \mathbb{N}}$ denoting the set of all its non-negative eigenvalues being repeated the same number of times as its multiplicity. Then $T \in S_1(\mathcal{H})$ if and only if*

$$\sum_{n \in \mathbb{N}} |\lambda_n| < +\infty$$

and

$$\operatorname{Tr} T = \sum_{n \in \mathbb{N}} \lambda_n,$$

where $\operatorname{Tr} T$ denotes the trace of T.

We write the trace norm in terms of the trace as

$$\|T\|_1 = \operatorname{Tr}\left(\sqrt{T^*T}\right).$$

We end with a result on the trace norm of an integral operator.

Proposition A.4. *Let $Q \subset \mathbb{R}^d$ be a cube and let $K \in L^2(Q)$ be an integral operator with an integral kernel $K(x, y)$. Let ℓ be the smallest even integer such that $l > d/2$. Assume that*

$$M = \max_{|\alpha| \le \ell} \int_Q \int_Q |\partial_x^\alpha k(x, y)|^2 \mathrm{d}x\mathrm{d}y < +\infty. \tag{A.3}$$

Then $K \in S_1$ and

$$\|K\|_{S_1} \le C(Q)M, \tag{A.4}$$

where $C(Q)$ depends only on the dimension d and on the size of the cube Q.

Proof. Consider the operator $-\Delta$ in $L^2(Q)$ with the Dirichlet boundary conditions on the boundary of Q. This operator can be explicitly diagonalised by Fourier series. As a consequence, it is easy to establish that $(-\Delta)^{-\ell/2} \in S_2$ and the norm $\|(-\Delta)^{-\ell/2}\|_2$ depends only on d and the size of Q. Now write

$$K = (-\Delta)^{-\ell/2}(-\Delta)^{\ell/2}K.$$

By (A.3), the operator $(-\Delta)^{-\ell/2}$ belongs to S_2 and further we have the estimate

$$\|(-\Delta)^{\ell/2}K\|_{S_2} \le C(d)M.$$

This completes the claim. □

Remark A.5. One can make this proposition sharper by considering non-integer values of ℓ; then (A.3) has to be formulated in terms of Sobolev spaces.

Corollary A.6. *Let $Q \subset \mathbb{R}^d$ be a cube and let $K \in L^2(Q)$ be an integral operator with an integral kernel $K(x, y)$. Let ℓ be the smallest even integer such that $\ell > d/2$. Assume that $C^\ell(Q \times Q)$. Then $K \in S_1$ and*

$$\|K\|_{S_1} \le C(Q)\|K\|_{C^\ell}.$$

Remark A.7. By using a smooth atlas and a partition of unity, the above corollary can be extended to the case when Q is a smooth compact manifold, where $\dim Q = d$.

Appendix B. Symbol calculus

We here define the symbol calculus necessary for Section 4. We begin by defining the class $S^m = S^m(T^*\mathbb{S}^{d-1})$ as the set containing functions $f \in C^\infty(T^*\mathbb{S}^{d-1})$ satisfying the estimates

$$|\partial_x^\alpha \partial_\omega^\beta f(x, \omega)| \le C_{\alpha\beta}^f \langle x\rangle^{m-|\alpha|}, \quad m \in \mathbb{R}, \tag{B.1}$$

for all multi-indices α and β, where $C^f_{\alpha\beta}$ is a constant depending only on α, β and f.

Lemma B.1. *If $f \in S^m(T^*\mathbb{S}^{d-1})$ and $g \in S^n(T^*\mathbb{S}^{d-1})$, then $fg \in S^{m+n}(T^*\mathbb{S}^{d-1})$.*

Proof. By using the product rule, it follows that

$$\partial^\beta_\omega (fg)(x,\omega) = \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1!\beta_2!}(\partial^{\beta_1}_\omega f)(\partial^{\beta_2}_\omega g).$$

By using the product rule again on the above equation, we obtain

$$\partial^\alpha_x \partial^\beta_\omega (fg)(x,\omega) = \sum_{\substack{\beta_1+\beta_2=\beta \\ \alpha_1+\alpha_2=\alpha}} \frac{\alpha!\beta!}{\alpha_1!\beta_1!\alpha_2!\beta_2!}(\partial^{\alpha_1}_x\partial^{\beta_1}_\omega f)(\partial^{\alpha_2}_x\partial^{\beta_2}_\omega g). \tag{B.2}$$

Applying estimate (B.1) to the two terms in (B.2) yields

$$|\partial^\alpha_x \partial^\beta_\omega (fg)(x,\omega)| \leq C^{fg}_{\alpha\beta}\langle x\rangle^{m-|\alpha_1|}\langle x\rangle^{n-|\alpha_2|} = C^{fg}_{\alpha\beta}\langle x\rangle^{(m+n)-|\alpha|}$$

where $C^{fg}_{\alpha\beta}$ is a polynomial combination of the constants $C^f_{\alpha\beta}$ and $C^g_{\alpha\beta}$. □

Lemma B.2. *If $f \in S^{-m}(T^*\mathbb{S}^{d-1})$ for $m > 0$ then $e^{if} \in S^0(T^*\mathbb{S}^{d-1})$.*

Proof. Firstly, note that in the case $\alpha = \beta = 0$ it is trivial that $e^{if} \in S^0(T^*\mathbb{S}^{d-1})$. So now suppose that $|\alpha + \beta| \geq 1$. Then by using the chain rule together with the product rule, it follows that

$$\partial^\beta_\omega e^{if(x,\omega)} = e^{if(x,\omega)} \sum_{(\beta_1+\cdots+\beta_{|\beta|}=\beta)} \prod_{i=1}^{|\beta|}(\partial^{\beta_i}_\omega f).$$

Repeating this procedure yields

$$\partial^\alpha_x\partial^\beta_\omega e^{if(x,\omega)} = e^{if(x,\omega)} \sum_{\substack{(\alpha_1+\cdots+\alpha_{|\alpha|}=\alpha) \\ (\beta_1+\cdots+\beta_{|\beta|}=\beta)}} \prod_{i=1}^{|\alpha+\beta|}(\partial^{\alpha_i}_x\partial^{\beta_i}_\omega f).$$

Since $f \in S^{-m}(T^*\mathbb{S}^{d-1})$ with $m > 0$, then

$$|\partial^\alpha_x\partial^\beta_\omega e^{if(x,\omega)}| \leq C^f_{\alpha\beta}\langle x\rangle^{-m-|\alpha|} \leq C^f_{\alpha\beta}\langle x\rangle^{-|\alpha|}$$

so that $e^{if} \in S^0(T^*\mathbb{S}^{d-1})$ as required. □

Lemma B.3. *If $f \in S^{-m}(T^*\mathbb{S}^{d-1})$ with $m > 0$ then $e^{if} - 1 \in S^{-m}(T^*\mathbb{S}^{d-1})$.*

Proof. Note first that for the case $\alpha = \beta = 0$, we have the estimate

$$|e^{if} - 1| \leq C|f| \leq C\langle x\rangle^{-m},$$

since $f \in S^{-m}(T^*\mathbb{S}^{d-1})$. Then for the case $|\alpha + \beta| \geq 1$, we may repeat the arguments of Lemma B.2 and the result follows. □

Appendix C. Proof of Proposition 2.4

C.1. **Semiclassical ΨDO in $\mathbb{R}^n$.** Let $S^m_{\mathrm{comp}}(\mathbb{R}^{3n})$ be the class of amplitudes $A = A(x, y, \xi)$ that are compactly supported in x and y and which satisfy the estimates

$$|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma A(x, y, \xi)| \leq C_{\alpha\beta\gamma}\langle\xi\rangle^{m-|\alpha|}$$

for multi-indices α, β and γ. This is a subset of a more general class defined in [23] Definition 23.3. In the class defined in [23], there is no assumption of compact support. For an amplitude function $A \in S^m_{\mathrm{comp}}(\mathbb{R}^{3n})$ we define the semiclassical pseudodifferential operator $\mathrm{Op}_h[A] : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ with semiclassical parameter $h > 0$ as

$$(\mathrm{Op}_h[A]u)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi/h} A(x, y, \xi)u(y)\mathrm{d}y\mathrm{d}\xi.$$

We now state various norm estimates for the operator $\mathrm{Op}_h[A]$ for $A \in S^m_{\mathrm{comp}}(\mathbb{R}^{3n})$. Firstly, for $m \leq 0$, the operator $\mathrm{Op}_h[A]$ is bounded (cf. [23]). We will only be interested in the case $m < 0$, and by the Calderon-Villancourt theorem (see e.g. [25]) combined with a scaling argument, we obtain the estimate

$$\sup_{0<h<1} \|\mathrm{Op}_h[A]\| \leq C(A).$$

Next, for $m < -n$ the operator $\mathrm{Op}_h[A]$ belongs to the trace class and the trace norm can be estimated by

$$\|\,\mathrm{Op}_h[A]\|_1 \leq Ch^{-n}, \quad m < -n.$$

This can be found for example in [23] Proposition A.2.4. Finally, by applying abstract interpolation to the two estimates above, we have that $\mathrm{Op}_h[A]$ belongs to the Schatten class S_p for $m < -\frac{n}{p}$ and

$$\|\,\mathrm{Op}_h[A]\|_p \leq Ch^{-n/p}, \quad m < -\frac{n}{p}. \tag{C.1}$$

Next, for $A \in S^m_{\mathrm{comp}}(\mathbb{R}^{3n})$ we define the left symbol A_L of A as

$$A_L(x, x', \xi) = A(x, x, \xi)$$

and the right symbol A_R of A as

$$A_R(x, x', \xi) = A(x', x', \xi).$$

Note that although the left and right symbols no longer enjoy the assumption of compact support, the operators $\mathrm{Op}_h[A_L]$ and $\mathrm{Op}_h[A_R]$ are still well defined, see for instance [23] Section 23. Further, the Schatten p-norm estimates (C.1) are still applicable to these operators; see for instance [23] Proposition A.2.3 for the operator norm estimate and [24] Section 3.4 for the trace class estimates.

Lemma C.1. *Let $A \in S^m_{comp}(\mathbb{R}^{3n})$, $m < 0$, and let $p \geq 1$ with $m < -\frac{n}{p}$. Then*

$$\|\,\mathrm{Op}_h[A] - \mathrm{Op}_h[A_{L,R}]\|_p = O(h^{-\frac{n}{p}+1}), \quad h \to +0,$$

where the above refers to two separate estimates, one for the left symbol and one for the right symbol.

Proof. By Taylor's formula,

$$A(x,x',\xi)-A(x,x,\xi)=\sum_{j=1}^{n}(x'_j-x_j)\int_0^1\frac{\partial}{\partial\nu_j}A(x,\nu,\xi)|_{\nu=x+t(x'-x)}\mathrm{d}t.$$

It follows from this formula that the integral kernel of $\mathrm{Op}_h[A]-\mathrm{Op}_h[A_L]$ can be expressed as

$$\begin{aligned}&\sum_{j=1}^{n}\int_0^1\int_{\mathbb{R}^n}(x'_j-x_j)e^{i(x-x')\cdot\xi/h}\frac{\partial}{\partial\nu_j}A(x,\nu,\xi)|_{\nu=x+t(x'-x)}\mathrm{d}\xi\mathrm{d}t\\&=ih\sum_{j=1}^{n}\int_0^1\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial\xi_j}e^{i(x-x')\cdot\xi/h}\right)\frac{\partial}{\partial\nu_j}A(x,\nu,\xi)|_{\nu=x+t(x'-x)}\mathrm{d}\xi\mathrm{d}t\\&=-ih\sum_{j=1}^{n}\int_0^1\int_{\mathbb{R}^n}e^{i(x-x')\cdot\xi/h}A_j(x,x+t(x'-x),\xi)\mathrm{d}\xi\mathrm{d}t,\end{aligned}$$

where

$$A_j(x,x+t(x'-x),\xi)=\frac{\partial^2}{\partial\nu_j\partial\xi_j}A(x,\nu,\xi)|_{\nu=x+t(x'-x)}.$$

The result follows by applying the estimate (C.1) to the above. □

Lemma C.2. *Let $A\in S^m_{comp}(\mathbb{R}^{3n})$, $m<0$, $B\in S^k_{comp}(\mathbb{R}^{3n})$, $k<0$ and let $C\in S^{m+k}_{comp}(\mathbb{R}^{3n})$ be such that*

$$A(x,x,\xi)B(x,x,\xi)=C(x,x,\xi).$$

Then for $p\geq 1$, $m+k<-\frac{n}{p}$, we have

$$\|\mathrm{Op}_h[A]\,\mathrm{Op}_h[B]-\mathrm{Op}_h[C]\|_p=O(h^{-\frac{n}{p}+1}),\quad h\to+0.$$

Proof. First let

$$D(x,x',\xi)=A(x,x,\xi)B(x',x',\xi).$$

Then by a direct calculation using the Fourier inversion formula (see e.g.[23], proof of Theorem 23.6)

$$\mathrm{Op}_h[A_L]\,\mathrm{Op}_h[B_R]=\mathrm{Op}_h[D].\tag{C.2}$$

Next, notice that $D_L=C_L$. Let r,q be such that $r\geq 1$, $q\geq 1$, with $r^{-1}+q^{-1}=p^{-1}$ and $m<-\frac{n}{r}$, $k<-\frac{n}{q}$. Then by (C.2),

$$\begin{aligned}&\mathrm{Op}_h[A]\,\mathrm{Op}_h[B]-\mathrm{Op}_h[C]=(\mathrm{Op}_h[A]-\mathrm{Op}_h[A_L])\,\mathrm{Op}_h[B]+\\&+\mathrm{Op}_h[A_L](\mathrm{Op}_h[B]-\mathrm{Op}_h[B_R])+(\mathrm{Op}_h[D]-\mathrm{Op}_h[D_L])+(\mathrm{Op}_h[C_L]-\mathrm{Op}_h[C]).\end{aligned}$$

Now, using the Hölder type inequality for Schatten norms (A.1), we get using Lemma C.1:

$$\begin{aligned}
&\|(\operatorname{Op}_h[A] - \operatorname{Op}_h[A_L]) \operatorname{Op}_h[B]\|_p \leq \| \operatorname{Op}_h[A] - \operatorname{Op}_h[A_L]\|_r \| \operatorname{Op}_h[B]\|_q \\
&\qquad = O(h^{-\frac{n}{r}+1}) O(h^{-\frac{n}{q}}) = O(h^{-\frac{n}{p}+1}), \\
&\| \operatorname{Op}_h[A_L](\operatorname{Op}_h[B] - \operatorname{Op}_h[B_R]\|_p \leq \| \operatorname{Op}_h[A_L]\|_r \| \operatorname{Op}_h[B] - \operatorname{Op}_h[B_R]\|_q \\
&\qquad = O(h^{-\frac{n}{r}}) O(h^{-\frac{n}{q}+1}) = O(h^{-\frac{n}{p}+1}), \\
&\| \operatorname{Op}_h[D] - \operatorname{Op}_h[D_L]\|_p = O(h^{-\frac{n}{p}+1}), \\
&\| \operatorname{Op}_h[C_L] - \operatorname{Op}_h[C]\|_p = O(h^{-\frac{n}{p}+1}).
\end{aligned}$$

Combining the above four estimates gives the required result. □

C.2. **Change of variables.** We now describe a change of variables for a pseudodifferential operator $\operatorname{Op}_h[a]$ with symbol $a \in S^m(T^*\mathbb{S}^{d-1})$ with $m < 0$, to an operator $\operatorname{Op}_h[A]$ with $A \in S^m_{\text{comp}}(\Lambda_\nu \times \Lambda_\nu \times \Lambda_\nu)$ for $\nu \in \mathbb{S}^{d-1}$.

Fix $\nu \in \mathbb{S}^{d-1}$; we will denote by $\phi_\nu : \mathbb{R}^n \to \Lambda_\nu$ the orthogonal projection onto the plane Λ_ν. Explicitly,

$$\phi_\nu(x) = x - \langle x, \nu \rangle \nu, \quad x \in \mathbb{R}^d.$$

Next, consider the hemispherical domain

$$H_\nu = \{\omega \in \mathbb{S}^{d-1} : \langle \omega, \nu \rangle > 1/2\}.$$

We denote $Y_\nu = \phi_\nu(H_\nu)$; the set Y_ν forms a ball in Λ_ν with the radius $\sqrt{3}/2$ centered at the origin. The map ϕ_ν restricted to H_ν generates a unitary operator $U_\nu : L^2(H_\nu) \to L^2(Y_\nu)$,

$$(U_\nu f)(x) = \langle \nu, \omega \rangle^{-1/2} f(\omega), \quad x = \phi_\nu(\omega).$$

Let $a \in S^m(T^*\mathbb{S}^{d-1})$ with $m < 0$. Assume that $a(\omega, \xi) = 0$ for ω outside a 'small' compact set Ω with $\Omega \subset H_\nu$ for some $\nu \in \mathbb{S}^{d-1}$. Let $\chi \in C_0^\infty(\mathbb{S}^{d-1})$ be a function with support in H_ν such that $\chi = 1$ on Ω. Then, by using repeated integration by parts, one can check that the operator $\operatorname{Op}_h[a](1-\chi)$ has a smooth kernel and all partial derivatives of this kernel are $O(h^\infty)$. Thus, by the trace norm estimate for integral operators (A.4),

$$\| \operatorname{Op}_h[a](1 - \chi)\|_1 = O(h^\infty), \quad h \to 0.$$

This reduces our consideration to $\operatorname{Op}_h[a]\chi$. By an explicit calculation, one checks that

$$U_\nu \operatorname{Op}_h[a] \chi U_\nu^* = \operatorname{Op}_h[A], \tag{C.3}$$

where $A \in S^m_{\text{comp}}(\Lambda_\nu \times \Lambda_\nu \times \Lambda_\nu)$ and

$$A(x, x, \xi) = a(\omega, \eta), \quad x = \phi_\nu(\omega), \quad \xi = \phi_\omega(\eta). \tag{C.4}$$

Proof of Proposition 2.2. Let $\{\mu_j\}$ be a sufficiently fine smooth partition of unity on the sphere, that is

$$1 = \sum_j \mu_j.$$

Writing

$$a(\omega, \xi) = \sum_j \mu_j(\omega) a(\omega, \xi),$$
$$b(\omega, \xi) = \sum_j \mu_j(\omega) b(\omega, \xi),$$

we can reduce the question to the case when both a and b are supported on some hemispherical domain H_ν. Now, as above we can choose χ such that $\chi \in C_0^\infty(H_\nu)$ and $\chi = 1$ on the support of a and b. With an $O(h^\infty)$ error, we can replace $\operatorname{Op}_h[a]$ by $\operatorname{Op}_h[a]\chi$, $\operatorname{Op}_h[b]$ by $\operatorname{Op}_h[b]\chi$ and $\operatorname{Op}_h[ab]$ by $\operatorname{Op}_h[ab]\chi$. So we need to prove

$$\|(\operatorname{Op}_h[a]\chi)(\operatorname{Op}_h[b]\chi) - (\operatorname{Op}_h[ab]\chi)\|_p = O(h^{-\frac{d-1}{p}+1}), \quad h \to 0.$$

This is equivalent to

$$\|(U_\nu \operatorname{Op}_h[a]\chi U_\nu^*)(U_\nu \operatorname{Op}_h[b]\chi U_\nu^*) - (U_\nu \operatorname{Op}_h[ab]\chi U_\nu^*)\|_p = O(h^{-\frac{d-1}{p}+1}), \quad h \to 0.$$

By the explicit formulas (C.3) and (C.4), this follows from Lemma C.2. □

Proof of Proposition 2.3. Clearly, $\operatorname{Op}_h[a]^*$ has the right symbol $\bar{a}$. Following the procedure of change of variables in Section C.2 with obvious modifications, we obtain

$$\|(1-\chi)\operatorname{Op}_h[a]^*\|_1 = O(h^\infty), \quad h \to 0,$$
$$U_\nu \chi \operatorname{Op}_h[a]^* U_\nu = \operatorname{Op}_h[A_1],$$

where $A_1 \in S^m_{\mathrm{comp}}(\Lambda_\nu \times \Lambda_\nu \times \Lambda_\nu)$ and

$$A_1(x, x, \xi) = \overline{A(x, x, \xi)}.$$

Thus by Lemma C.1,

$$\|\operatorname{Op}_h[A_1] - \operatorname{Op}_h[\bar{A}]\|_p = O(h^{-\frac{d-1}{p}+1}), \quad h \to 0,$$

and the result follows. □

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